15.1: Functions of Several Variables

Curvature Formulas	When to use
$K = \left\lvert \frac{dT}{ds} \right\rvert$	$\vec{r}(s)$ parameterized by arclength
$K = \frac{\lvert \vec{T}'(t) \rvert}{\lvert \vec{r}'(t) \rvert}$	$\lvert \vec{r}'(t) \rvert = \text{constant}$ $\vec{T} = \frac{\vec{r}'}{\lvert \vec{r}' \rvert}$
$K = \frac{\lvert \vec{r}' \times \vec{r}'' \rvert}{\lvert \vec{r}' \rvert^3}$	use in most cases $\lvert \vec{r}'(t) \rvert$ is a fcn of t

Example; $\vec{r}(t) = \langle t^2, t^2, \frac{1}{2}t^2 \rangle$ on $1 \le t \le 4$

Reparameterize by arclength

$$\vec{r}'(t) = \langle 2t, 2t, t \rangle$$

$$\lvert \vec{r}'(t) \rvert = 3t \neq 1$$ Need to reparameterize

$$s = \int_1^t \lvert \vec{r}'(u) \rvert du = \int_1^t 3u \, du$$

$$= \left[\frac{3u^2}{2} \right]_1^t = \frac{3}{2}\left[t^2 - 1 \right]$$

$$\boxed{s = \frac{3}{2}\left[t^2 - 1 \right]}$$ relationship between s and t

Limits on s:

$@ t = 1$ $\quad s = \frac{3}{2}\left[1^2 - 1\right] = 0$

$@ t = 4$ $\quad s = \frac{3}{2}\left[4^2 - 1\right] = \frac{3 \cdot 15}{2} = \frac{45}{2}$

Limits on s: $\quad$ @ $t=1$

@ $t=4$ $\quad s = \frac{3}{2}\left[4^2-1\right] = \frac{3\cdot 15}{2} = \frac{45}{2}$

$$0 \le s \le \frac{45}{2}$$

Solve for t as a fcn of s

$$s = \frac{3}{2}\left[t^2-1\right]$$

$$\frac{2}{3}s = t^2-1 \longrightarrow t = \pm\sqrt{\frac{2}{3}s+1}$$

Reparameterization

$$\boxed{\tilde{\vec{r}}(s) = \vec{r}\left(\pm\sqrt{\frac{2}{3}s+1}\right) = \left\langle \frac{2}{3}s+1,\ \frac{2}{3}s+1,\ \frac{1}{2}\left(\frac{2}{3}s+1\right)\right\rangle \quad 0 \le s \le \frac{45}{2}}$$

$\tilde{\vec{r}}(s) \longrightarrow$ length from $\tilde{\vec{r}}(0)$ to $\tilde{\vec{r}}(s)$ is s

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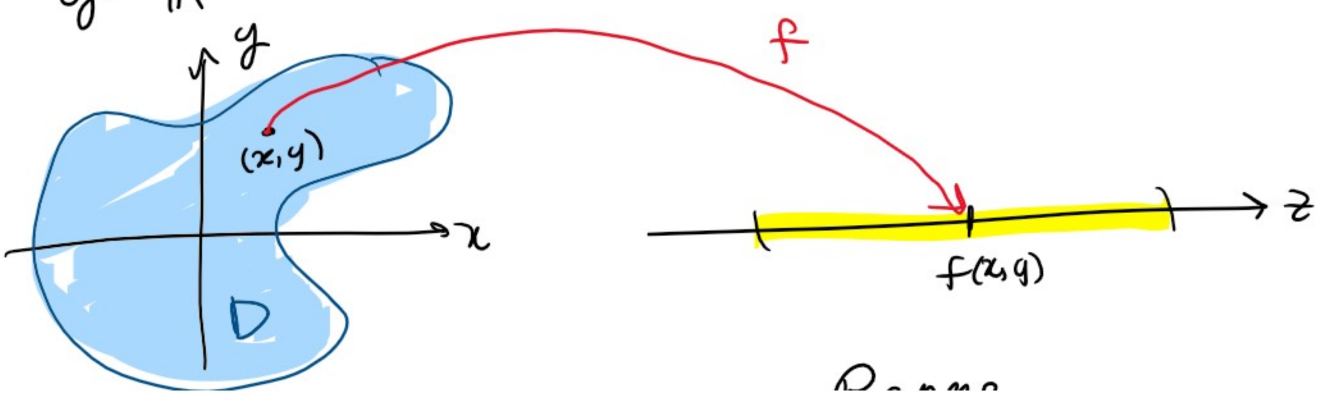
Functions of Several Variables

Def: A <u>function of two variables</u> $z=f(x,y)$ maps each ordered pair (x,y) in a subset D of $\mathbb{R}^2$ to a real number z.



D - Domain

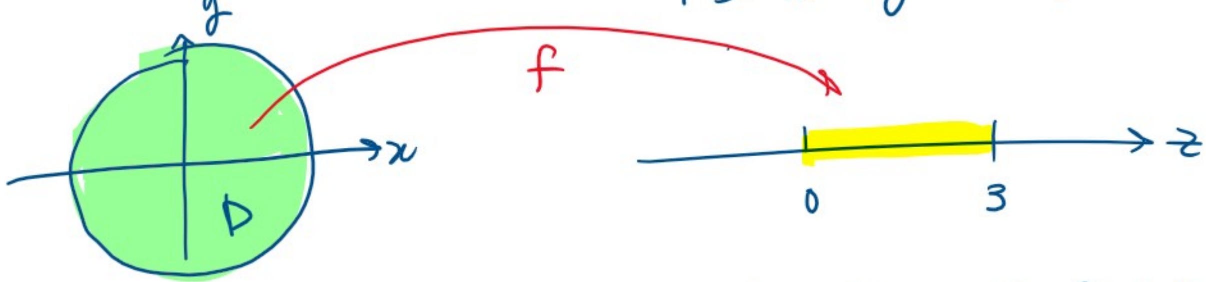
Range

The Range is set of z that satisfy $z = f(x,y)$ for every (x,y) in D

Example: $z = f(x,y) = \sqrt{9 - x^2 - y^2}$

Domain: Need $9 - x^2 - y^2 \geq 0$

$9 \geq x^2 + y^2$ disk of radius 3.



Domain $0 \leq x^2 + y^2 \leq 9$

Range:

$$0 \leq \sqrt{9 - x^2 - y^2} \leq 3$$

Range $[0, 3]$

We've seen Functions of two variables

Plane: $x + 2y + 3z = 1$

$\longrightarrow z = \frac{1}{3}(1 - x - 2y) = f(x,y)$

Quadric Surfaces

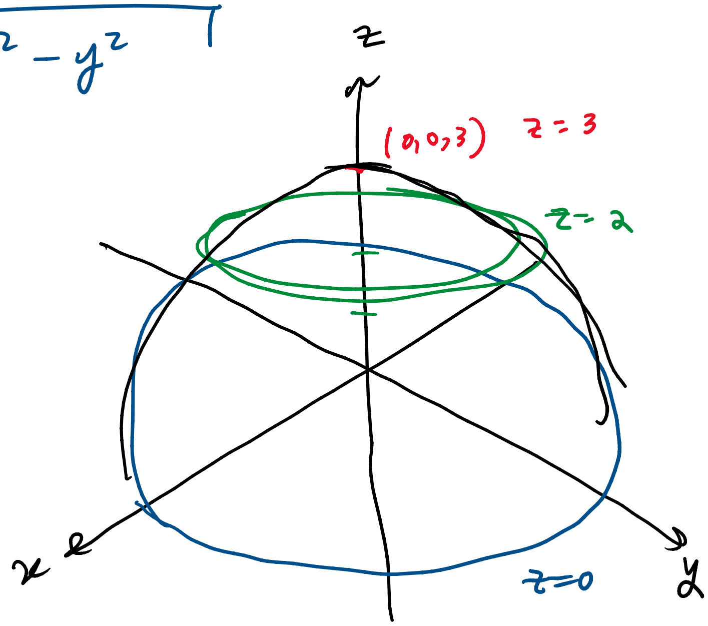
$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Elliptic Parabola

$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Parabolic Hyperboloid

Vector-Valued Functions	Functions of Two Variables
$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$	$z = f(x,y)$
Input: t – one indep. var.	Input: x and y (2 indep vars)
Output: $\vec{r}$ – vector (3 dimensions)	Output: z – scalar (1 dim)
graph: curve (1 dim.)	graph: surface (2 dim)

Graphing: Look at traces $z = C$

Example: $z = \sqrt{9 - x^2 - y^2}$



$\boxed{z=0}$ $\left(0 = \sqrt{9 - x^2 - y^2}\right)^2$

$0 = 9 - x^2 - y^2$

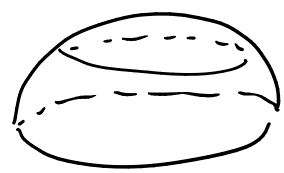
$x^2 + y^2 = 9$

circle of radius 3

$\boxed{z=2}$ $\left(2 = \sqrt{9 - x^2 - y^2}\right)^2$

$x^2 + y^2 = 9 - 4 = 5$

circle of radius $\sqrt{5}$



top half of sphere of radius 3

$\boxed{z=3}$ $\left(3 = \sqrt{9 - x^2 - y^2}\right)^2$

$x^2 + y^2 = 0$

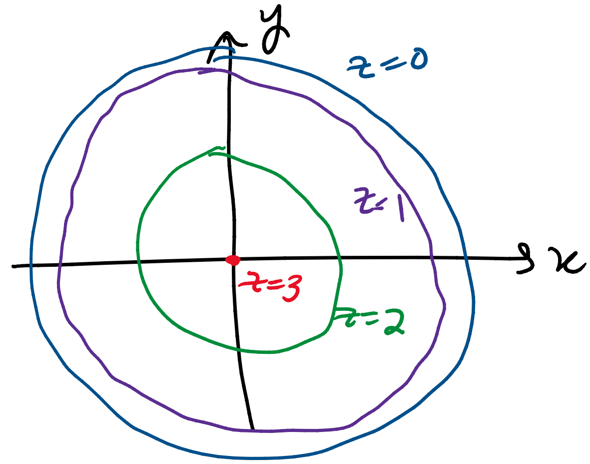
$\boxed{z=3}$ $3=\sqrt{9-x^2-y^2}$

$x^2+y^2=0$

$(0,0)$

of radius 3

We can condense these traces and plot all in the xy plane



<u>Def</u>: Given $f(x,y)$ and a member c in the range, a <u>level curve</u> for the value c is the set of points that satisfy $f(x,y)=c$

$c=2$ level curve $x^2+y^2=5$

<u>Topographical Maps</u> - level curve represents constant elevation

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<u>Example</u>: Sketch the surface

$$z=\cos(xy)$$

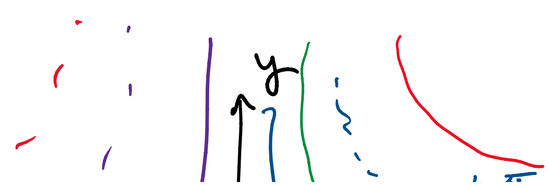
<u>Domain</u>: $\cos(\theta)$ $\quad -\infty<\theta<\infty$

$-\infty<xy<\infty$

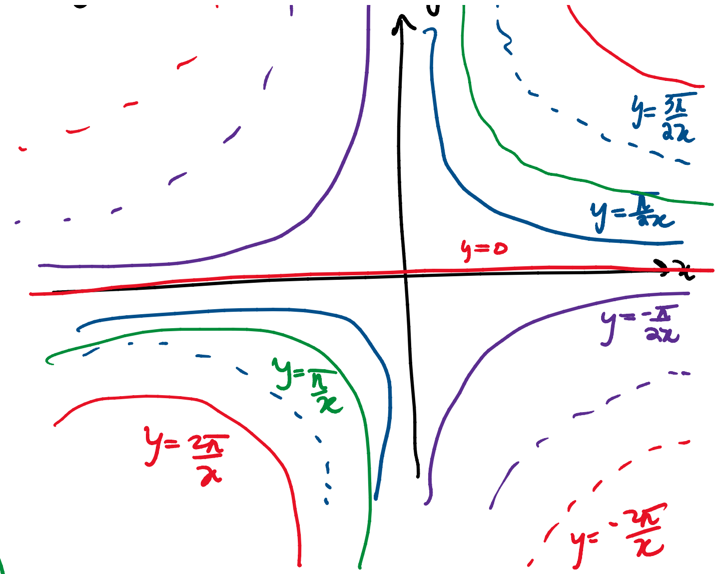
$-\infty<x<\infty$ and $-\infty<y<\infty$

<u>Range</u>: $[-1,1]$

<u>Level Curves</u>: $z=c=\cos(xy)$



Level curves:



$\boxed{c=0}$

$0 = \cos(xy) = \cos(\theta)$

$\theta = \frac{\pi}{2}, \frac{-\pi}{2}, \frac{3\pi}{2}, \ldots$

$xy = \frac{\pi}{2}, \frac{-\pi}{2}, \frac{3\pi}{2}, \ldots$

$y = \frac{\pi}{2x}, \frac{-\pi}{2x}, \frac{3\pi}{2x}, -\frac{3\pi}{2x}$

$\boxed{c=1}$ $\quad 1 = \cos(xy) = \cos\theta$

$\theta = 0, 2\pi, 4\pi, \ldots$

$xy = 0, 2\pi, 4\pi, \ldots$

$y = 0, \frac{2\pi}{x}$

$\boxed{c=-1}$

$-1 = \cos(xy)$

$xy = \pi, 3\pi, 5\pi, \ldots$

$y = \frac{\pi}{x}$