Section 1.1

☆ Differential Equations & Mathematical Models

I. What is a ODE?

Def: An ordinary Differential Equation (ODE) is an equation that relates a function of one variable $y(t)$ with its derivatives

Ex: $\frac{dy}{dt} = y$ ① $\quad \frac{dy}{dt} = ty$ ② $\quad \frac{dy}{dt} = y^2$ ③

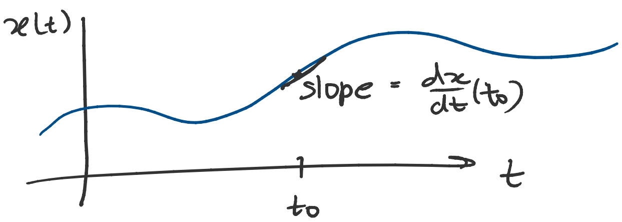
A Partial Differential Eqn (PDE) includes a function of multiple variables $u(t, x, y, z)$ and its partial derivatives.

Ex: $u_t = u_x \qquad u_t = u_{xx} + u_{yy}$

	Algebra	ODEs
Eqn:	$5x + 3 = 18$	$5\frac{dx}{dt} + 3x = 18$
Obj:	Find x	Find $x(t)$
Soln:	x is a single number ($x = 3$)	x is a function of the variable t

II. Why study ODEs?

- eqns whose solutions are functions



$5\frac{dx}{dt} + 3x = 18$

- here, derivative $\frac{dx}{dt}$ represents change
  - time
  - space
  - volume
  - $$$
  - concentration (chemical species)

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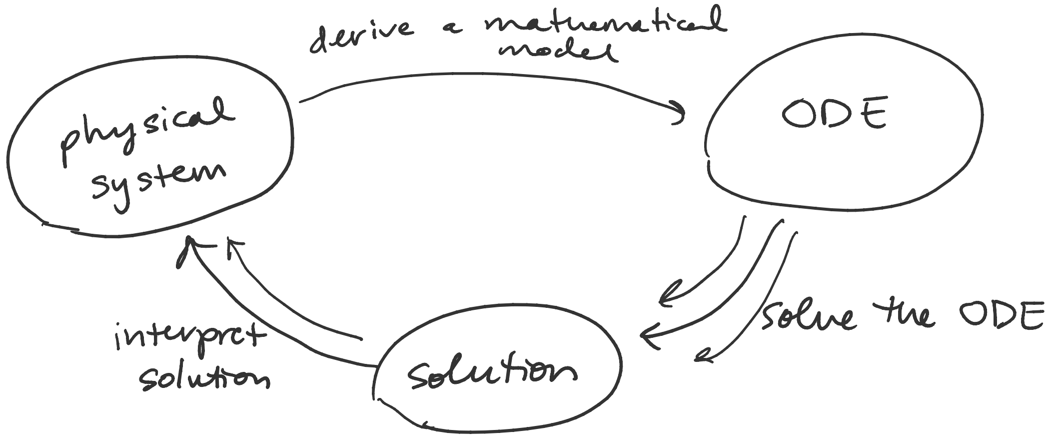

Examples:
- fluid mechanics
- control systems
- thermodynamics
- elasticity
- electro dynamics
- chemical reactions

Q: Why don't we just solve on computer?

1. Numerical solutions are approximations
   → need analytical solutions to estimate error

2. Human error + user error
   → need mathematical intuition to check solns.

GOAL: Develop mathematical intuition + conceptual understanding of ODEs.

III Types of ODEs

Def: The order of an ODE – highest order derivative in the eqn

1st order – contains $\frac{dy}{dt}$ and/or y, t $\qquad \frac{dy}{dt} = 5t + 3 + y$

2nd order – contain $\frac{d^2y}{dt^2}$ and/or $\frac{dy}{dt}, y, t$ $\qquad \frac{d^2y}{dt^2} = y + t$

3rd order – contain $\frac{d^3y}{dt^3}$ and/or lower order terms $\qquad \frac{d^3y}{dt^3} = t\frac{dy}{dt}$

Practice: Find the order of each ODE

$y\frac{dy}{dt} + 5 = 0$ — 1st order

$\left(\frac{dy}{dt}\right)^2 + ty = 0$ — 1st order $\left(\frac{dy}{dt}\right) = \left(\frac{dy}{dt}\right)\left(\frac{dy}{dt}\right)$

$\frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{d^4y}{dt^4} = 0$ — 4th order

$(1-y^2)\frac{d^2y}{dt^2} = 0$ — 2nd order

Linear vs Nonlinear

Def: A linear DE is any ODE that can be written in the form: (nth order)

$$c_n(t)\frac{d^ny}{dt^n} + c_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \dots + c_1(t)\frac{dy}{dt} + c_0(t)y = g(t)$$

A 1st order linear ODE

$$a(t)\frac{dy}{dt} + b(t)y = g(t)$$

Ex: $3\frac{dy}{dt} + 5y = t$

$t^2\frac{dy}{dt} + (5-t)y = \sin(t)$

A 2nd order linear ODE

$$a(t)\frac{d^2y}{dt^2} + b(t)\frac{dy}{dt} + c(t)y = g(t)$$

A nonlinear ODE cannot be written in the form above

Ex:

Linear	Nonlinear
$\frac{dy}{dt} + 3y = t$	$\frac{dy}{dt}(3-y) = t$
	$3\frac{dy}{dt} - \boxed{y\frac{dy}{dt}} = t$
$\frac{d^2y}{dt^2} - 2y = 0$	$\frac{d^2y}{dt^2} - \boxed{y^2} = 0$

Practice: Linear or Nonlinear $\left(y' = \frac{dy}{dt}\right)$

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$\underbrace{(1-t^2)}_{a(t)}y' + \underbrace{2t}_{b(t)}y - \underbrace{\sin(t)}_{g(t)} = 0$ linear

$y(1-t^2+y') = ty' = y - t^2y + \boxed{yy'}$ nonlinear

$e^t y' = \boxed{y^2} + 5$ nonlinear

$\frac{y''}{t^5} = \sqrt{t}\,y + \cos(t)$ linear

$y' = \boxed{\sin(y)}$ nonlinear

$y' = \sin(t)$ linear

$(1+t^2)y'' = 1$ linear

IV. IVPs

A typical problem is called an initial value problem (IVP)

$$\frac{dy}{dx} = f(x,y) \qquad \underbrace{y(x_0) = y_0}_{\text{initial condition}}$$

A solution must satisfy both conditions

Ex: $\frac{dy}{dt} = -y \qquad y(0) = 7$

Suppose I tell you the solution is

$$y(t) = 7e^{-t}$$

Verify this is true

1. Plug $y(t)$ into ODE
2. Plug $t=0$ into $y(t)$ check initial condition

→ 1. ODE: $\frac{dy}{dt} = -y$

Calculate $\frac{dy}{dt} = \frac{d}{dt}(7e^{-t}) = -7e^{-t}$

$-7e^{-t} = -(7e^{-t})$ ✓

→ 2. Plug $t=0$ into $y(t)$ to check init. cond

$y(0) = 7$

$y(0) = \left[7e^{-t}\right]\Big|_{t=0} = 7e^{-0} = 7$ ✓

So yes, $y(t) = 7e^{-t}$ solves the IVP

<u>Practice</u>: Assume $y(t) = \cos(3t) + C$ solves the IVP

$\frac{dy}{dt} = -3\sin(3t)$ $\quad\quad y(0) = 5$

Find the coefficient C so that $y(t)$ solves the IVP

1. Plug $y(t)$ into ODE

$\frac{dy}{dt} = -3\sin(3t)$

$\frac{d}{dt}\left[\cos(3t) + C\right] = -3\sin(3t)$

$-3\sin(3t) + 0 = -3\sin(3t)$ ✓

2. check initial condition

$y(0) = 5$

$\left[\cos(3t) + C\right]\Big|_{t=0} = 5$

$\cos(0) + C = 5$

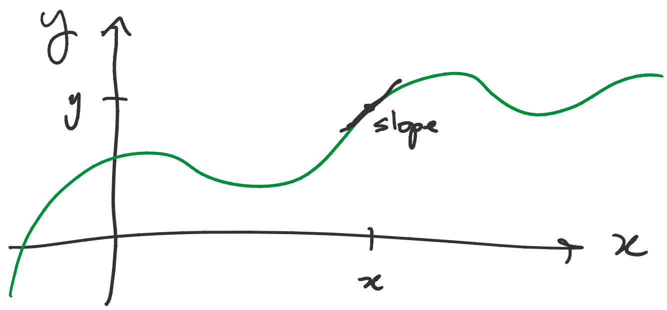
$$\cos(0) + C = 5$$
$$1 + C = 5$$
$$\boxed{C = 4}$$

So the solution is $\boxed{y(t) = \cos(3t) + 4}$

Ex: Sec 1.1 # 27

Write a differential equation of the form
$$\frac{dy}{dx} = f(x,y)$$
which has the fcn g as its solution

The slope of the graph g at the point (x,y) is the sum of x and y



$$\frac{dy}{dx} = \frac{dg}{dx} = \underset{(x,y)}{\underset{@}{\text{slope}}} = x + y \qquad \text{has soln } g$$

so the ODE is $\boxed{\frac{dy}{dx} = x + y}$

Ex: Sec 1.1 # 32

Write a differential equation that is a mathematical model of the situation described

The time rate of change of a population P is proportional to the square root of P

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$\frac{dP}{dt}$ is proportional to $\sqrt{P}$

$$\boxed{\frac{dP}{dt} = k\sqrt{P}}$$

proportionality constant