Section 1.2

## ☆ Integrals as General & Particular Solutions:

**Warmup:** List as many integration rules as you can

( power rule, u-substitution, integration by parts
trig substitution, trig fcns, $\frac{1}{x}$, $e^x$, $\ln x$
Review these methods as necessary )

**Today:** Simple ODE:

$$(*) \qquad \frac{dy}{dx} = \underbrace{f(x)}_{\text{RHS only depends on } x \text{, no } y \text{ terms}}$$

linear
1st order

Solve by integrating both sides wrt $x$

$$\int \frac{dy}{dx}\,dx = \int f(x)\,dx$$

$$y(x) = \int f(x)\,dx + C$$

This is the general solution to $(*)$

**Ex:** Consider the IVP:

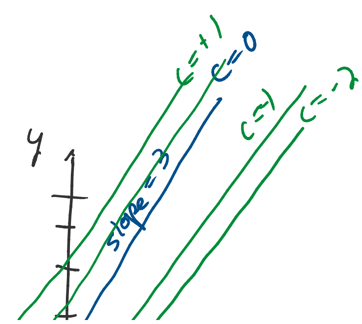
$$\left\{ \frac{dy}{dx} = 3 \quad , \quad y(0) = 1 \right\}$$

1. Solve ODE
2. Satisfy the initial condition

1. Solve the ODE $\longrightarrow$ integrate

$$\int \frac{dy}{dx}\,dx = \int 3\,dx$$

$$+ C$$

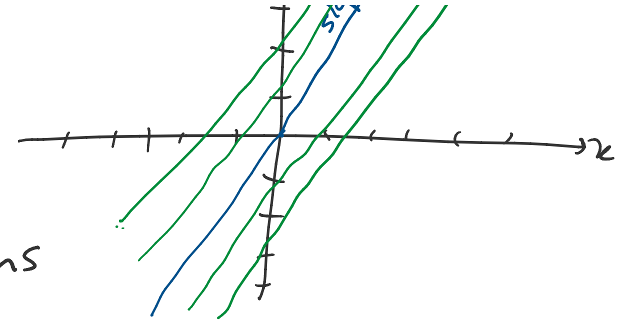


$$\int \frac{dy}{dx}\, dx = \int 3\, dx$$

$$y(x) = 3x + C$$

general solution

This gives a <u>family</u> of solutions




2. Satisfy the initial condition

$$y(0) = 1 = \left[3x + C\right]\Big|_{x=0} = C \rightarrow \boxed{C = 1}$$

solution to IVP: $\boxed{y(x) = 3x + 1}$

called a <u>particular</u> <u>solution</u>

- describes a single solution

<u>Q</u>: What happens if we have a 2nd order ODE?

$$\frac{d^2y}{dx^2} = 2x \qquad \leftarrow$$

RHS depends only on $x$ (no $y$ or $\frac{dy}{dx}$)

→ Integrate twice!

$$\int \frac{d^2y}{dx^2}\, dx = \int 2x\, dx$$

$$\frac{dy}{dx} = \frac{2x^2}{2} + C_1$$

→ integrate again

$$\int \frac{dy}{dx}\, dx = \int x^2 + C_1 \; dx$$

$$\boxed{y(x) = \frac{x^3}{3} + C_1 x + C_2}$$

general solution

<u>Note</u>: 2 unknown parameters $C_1$ and $C_2$

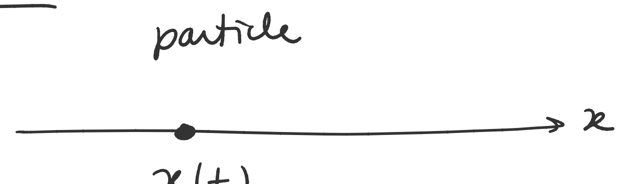
... family of solns

Note: 2 unknown parameters $c_1$, $c_2$
called a <u>2-parameter family of solns</u>

→ Need 2 initial conditions to select $c_1$ and $c_2$ to get a particular solution.

## II. Velocity - Acceleration Models

particle moving in $x$-direction




$x(t)$ - position

$v(t) = \frac{dx}{dt}$ velocity

$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ acceleration

<u>Newton's 2nd Law of Motion:</u>

Force $F(t)$ acts on particle (in $x$-direction)
$m$ - mass of particle

$$m\,a(t) = F(t)$$

If the force $F(t)$ is known

$$x'' = a(t) = \frac{F(t)}{m}$$

$$\boxed{x'' = \frac{F(t)}{m}}$$ is an ODE!

<u>Ex</u>: (constant acceleration)
Assume $F(t) = F$ is constant
then $a = F/m$ is also constant

2nd order linear ODE

IVP: $\begin{cases} x'' = a \\ x(0) = x_0 \end{cases}$

IVP: $\begin{cases} x'' = a \\ \underbrace{v(0) = v_0}_{\text{initial velocity}} \qquad \underbrace{x(0) = x_0}_{\text{initial position}} \end{cases}$

1. Solve the ODE $\longrightarrow$ integrating twice
2. Satisfy the initial conditions

$\rightarrow$ 1.

$$\int x'' \, dt = \int a \, dt$$

$$v = x' = at + c_1$$

$$\int x' \, dt = \int at + c_1 \, dt$$

$$x(t) = \frac{at^2}{2} + c_1 t + c_2$$

general solution

$\rightarrow$ 2. satisfy initial conditions

$$v(0) = v_0 = \left[ at + c_1 \right]\Big|_{t=0} = c_1$$

$$\rightarrow \boxed{c_1 = v_0}$$

$$x(0) = x_0 = \left[ \frac{at^2}{2} + c_1 t + c_2 \right]\Big|_{t=0} = c_2$$

$$\boxed{c_2 = x_0}$$

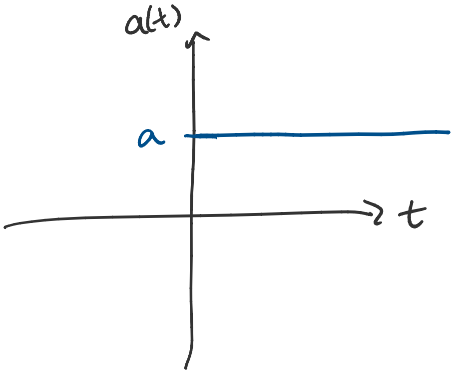
Particular solution is

$$\boxed{x(t) = \tfrac{1}{2} a t^2 + v_0 t + x_0}$$

Note: when $a$ is constant, we know $v(t)$ and $x(t)$ for all time $t$

acceleration velocity position

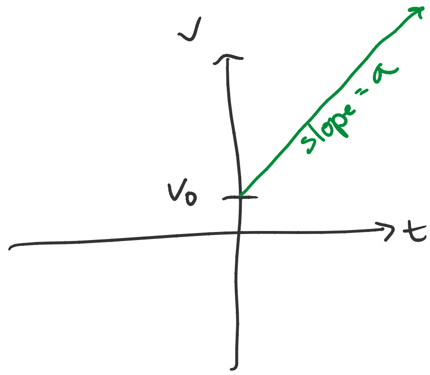
acceleration




$a(t) = a$

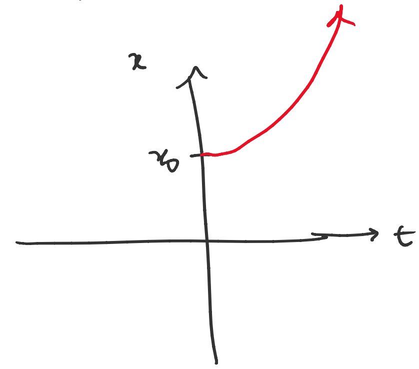
constant

velocity




$v(t) = at + v_0$

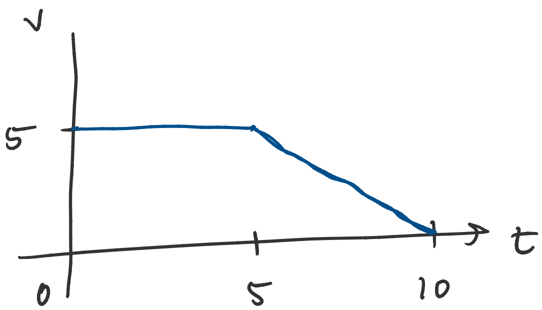
linear




$x(t) = \frac{1}{2}at^2 + v_0 t + x_0$

quadratic

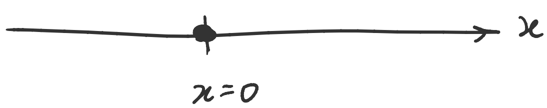
Ex: A particle travels with the following velocity




Draw the position of the particle $x(t)$

Assume that $x(0) = 0$

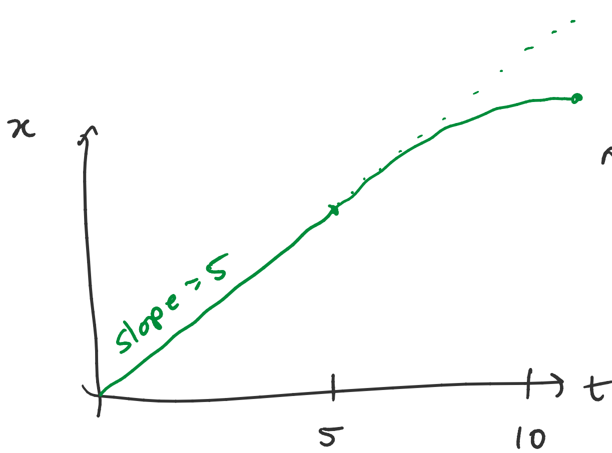
Remember $x(t) = \int v(t)\,dt$




$0 \leq t \leq 5$ particle moves right with vel = 5

$5 < t \leq 10$ particle slows down

at $t = 10$ vel = 0




quadratic curve underneath the line w/ slope = 5.

Ex: Consider a moving particle with

$a(t) = \cos(t)$ $v_0 = 1$ $x_0 = -1$

Find the position $x(t)$

2nd order, linear
RHS depends on $t$

IVP: $\begin{cases} x'' = \cos(t) \\ x'(0) = 1 \end{cases}$ $\quad x(0) = -1$

1. Solve the ODE $\to$ integrating $\times 2$
2. satisfy the initial conditions

$\to$ 1.

$$\int x''\,dt = \int \cos(t)\,dt$$

$$x' = \sin(t) + C_1 \quad \to \text{velocity}$$

$$\int x'\,dt = \int \sin(t) + C_1\,dt$$

$$x(t) = -\cos(t) + C_1 t + C_2$$

general solution

$\to$ 2. initial conditions

$$x'(0) = 1 = \left[\sin(t) + C_1\right]\Big|_{t=0} = C_1$$

$$\to \boxed{C_1 = 1}$$

$$x(0) = -1 = \left[-\cos(t) + C_1 t + C_2\right]\Big|_{t=0} = -1 + C_2$$

$$\boxed{C_2 = 0}$$

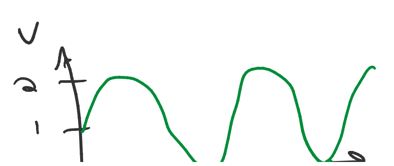
particular solution

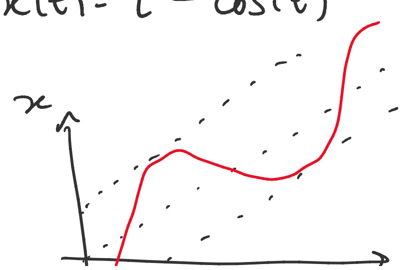
$$x(t) = -\cos(t) + t = \boxed{t - \cos(t)}$$

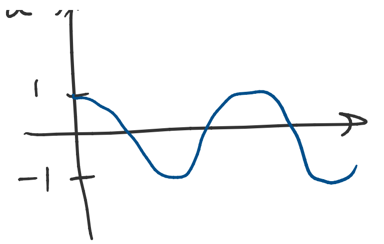
So we have:

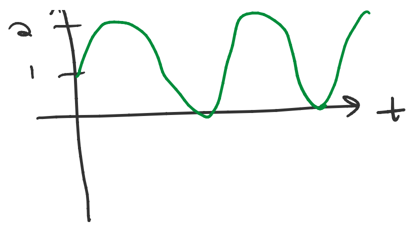
$a(t) = \cos(t)$ $\qquad$ $v(t) = \sin(t) + 1$ $\qquad$ $x(t) = t - \cos(t)$

plot these





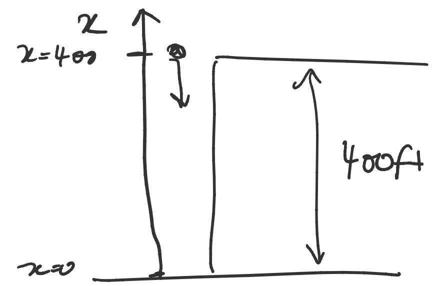
$x(t)$ oscillates with amplitude 1 around $x = t$

Ex: See 1.2 # 24

A ball is dropped from the top of a 400ft bldg

(a) How long does it take to reach the ground?

(b) With what speed does it strike the ground?




acceleration $a = g = -32\ ft/s^2$

$(g = -9.8\ m/s^2)$

$v_0 = 0$ (dropped from rest)

$x_0 = 400$

IVP: $\begin{cases} x'' = -32 \\ x'(0) = 0 \end{cases}$ $x(0) = 400$

1. Integrate twice

$$\int x''\,dt = \int -32\,dt$$

$$v = x' = -32t + c_1 \qquad \text{velocity}$$

$$\int x'\,dt = \int -32t + c_1\,dt$$

$$x(t) = -\frac{32t^2}{2} + c_1 t + c_2$$

general solution

2. Satisfy the initial conditions

$$x'(0) = 0 = \left[-32t + c_1\right]\Big|_{t=0} = \boxed{c_1 = 0}$$

$$x'(0) = 0 = \left[-32t + c_1\right]\Big|_{t=0} = \boxed{c_1 = 0}$$

$$x(0) = 400 = \left[-16t^2 + c_1 t + c_2\right]\Big|_{t=0} = \boxed{c_2 = 400}$$

particular solution

$$\boxed{x(t) = -16t^2 + 400}$$

(a) How long until the ball hits the ground?

find $t$ so that $x(t) = 0$

$$x(t) = -16t^2 + 400 = 0$$

$$400 = 16t^2$$

$$25 = t^2$$

$$\boxed{t = 5\,s}$$

(b) With what speed does the ball strike the ground?

find $v$ at $t=5$

$$v(t) = -32t$$

$$v = \left[-32t\right]\Big|_{t=5} = \boxed{-160\ ft/s}$$

So far we have studied

| Type | Eqn | Solution methods |
|---|---|---|
| 1st order linear (only $x$ terms on RHS) | $\frac{dy}{dx} = f(x)$ | direct integration $y(x) = \int f(x)\,dx + C$ |
| 2nd order linear | $\frac{d^2y}{dx^2} = f(x)$ | integrate twice |

| | | |
|---|---|---|
| 2nd order linear (only $x$ terms RHS) | $\frac{d^2y}{dx^2} = f(x)$ | twice |