✩ Slope Fields and Solution Curves:

Warm up: Find the general solution to the ODE

$$\frac{dy}{dx} = x(x-1) = x^2 - x$$

integrate both sides

$$\int \frac{dy}{dx}\, dx = \int (x^2 - x)\, dx$$

$$\boxed{y(x) = \frac{x^3}{3} - \frac{x^2}{2} + C}$$

I. Graphical Solutions

Consider the eqn: $\dfrac{dy}{dx} = f(x,y)$ $\quad (*)$ 1st order linear or nonlinear

A slope field is constructed by evaluating f at each point of a rectangular grid. At point (x,y) draw a line whose slope $= f(x,y)$

(Also called direction field)

equilibrium solution: y_* is an equilibrium solution for $(*)$ if

$$\frac{dy_*}{dx} = 0 \quad \text{for all } x \implies f(x, y_*) = 0 \text{ for all } x$$

Ex: $\dfrac{dy}{dx} = 10 - 2y$

1. Find an equilibrium soln
2. Draw a slope field

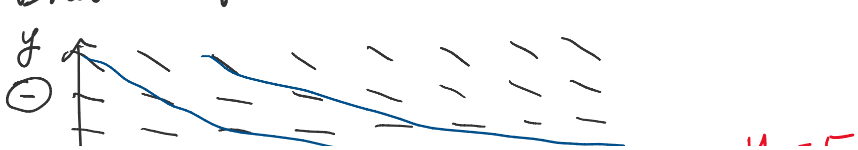
1. Equilibrium soln

$$\frac{dy_*}{dx} = 0 = 10 - 2y_*$$

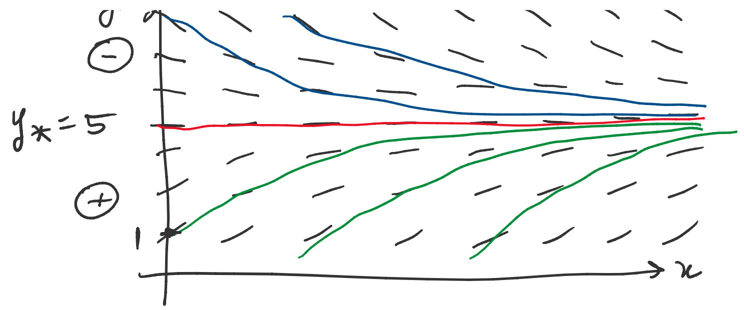
$$2y_* = 10$$

$\boxed{y_* = 5}$ equilibrium soln

2. Draw slope field



Gives us a family of solutions



$y_* = 5$

family of solutions

each soln is determind by its value at $x = 0$ (initial condition)

@ $y_* = 5$ → slope is zero → flat line

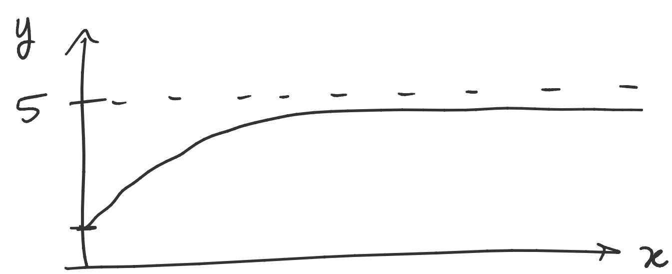
for $y > 5$, $f(x,y) = 10 - 2y < 0$
so $\frac{dy}{dx} < 0$ → negative slope ⊖
draw downward pointing lines

for $y < 5$ $f(x,y) = 10 - 2y > 0$
so $\frac{dy}{dx} > 0$ → positive slope ⊕
draw upward pointing lines

Draw solution curves - @ each (x, y) the slope of the solution curve must match the lines drawn

ex: $\frac{dy}{dx} = 10 - 2y \qquad y(0) = 1$

From the slope field, we know that the solution looks like



see that $\lim_{x \to \infty} y(x) = 5$

as $x \to \infty$, $y \to y_*$

Ex: $\frac{dy}{dx} = y + x$ 1st order linear $\left(\frac{dy}{dx} - y = x\right)$

(a) Find the equilibrium soln

(b) Draw a slope field

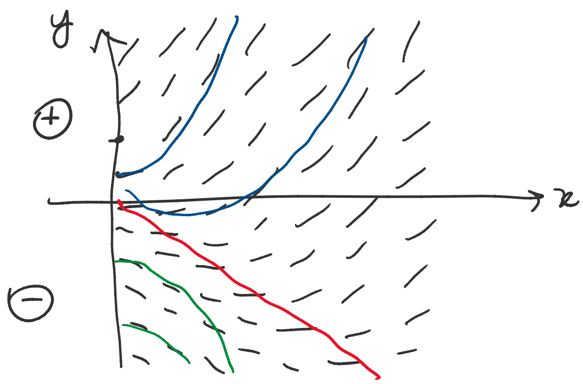
(c) What does the solution curve going through the point $(0, 2)$ look like?

(c) What does the solution curve going through the point (0, 2) look like?

(a) Eq soln: $\frac{dy_*}{dx} = 0 = y_* + x$

$\Rightarrow \boxed{y_* = -x}$ eq. soln is a fcn of x

(b) Slope field



1. Draw $y_* = -x$

   slope $= 0$ on $y_* = -x$

2. for $y > -x$

   $\frac{dy}{dx} = y + x > 0$ ⊕

3. for $y < -x$

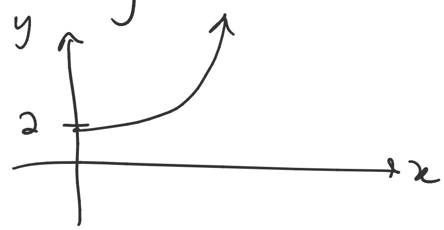
   $\frac{dy}{dx} = y + x < 0$ ⊖

4. Draw soln curves

(c) Solution curve through (0, 2)

$y(0) = 2$

looks like



II. Existence & Uniqueness

For the IVP: $\frac{dy}{dx} = f(x, y)$ $\quad y(x_0) = y_0$

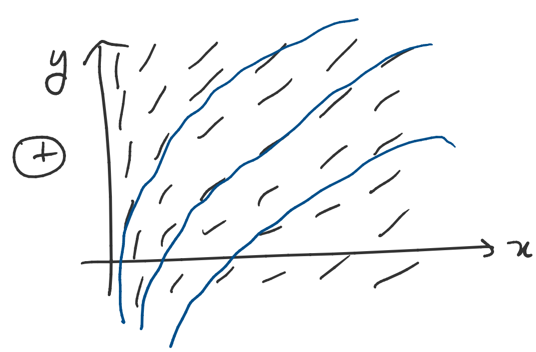
Q1: Do solutions exist?

Q2: If there is a solution, is it unique? or are there multiple possible solutions?

Counter example 1:

$\frac{dy}{dx} = \frac{1}{x}$ $\quad y(0) = 0$

eq soln: $\frac{dy_*}{dx} = 0 = \frac{1}{x}$ ✗



None of the solution curves go through (0, 0)

None of the solution curves pass through (0,0)

Solve by integrating

$$y(x) = \int \frac{1}{x}\,dx = \ln|x| + C$$

initial condition: $y(0) = 0 = \left[\underbrace{\ln|x|}_{\text{undefined } (-\infty)} + C\right]\Big|_{x=0}$ cannot solve

There is no solution that satisfies the IVP (solution does not exist)

Counter example 2

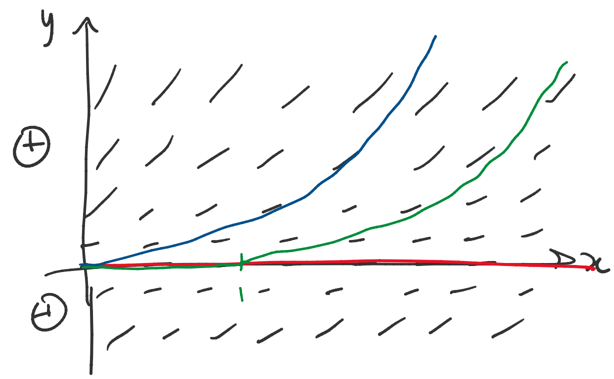
$$\frac{dy}{dx} = 3y^{2/3}, \quad y(0) = 0$$

eq. soln: $\frac{dy_*}{dx} = 0 = 3y_*^{2/3}$

$$\boxed{y_* = 0}$$

when $y>0$ $\quad \frac{dy}{dx} > 0$ $\oplus$

when $y<0$ $\quad \frac{dy}{dx} > 0$ $\oplus$



1. equilibrium solution $y_* = 0$ solves IVP
2. Note that $y_1(x) = x^3$ also solves the IVP

$$\frac{dy_1}{dx} = 3x^2 \overset{?}{=} 3y_1^{2/3} = 3\left[x^3\right]^{2/3} = 3x^2 \checkmark$$

so y_1 solves the ODE & IVP

3. Note $y_2(x) = \begin{cases} 0 & 0 \le x \le 1 \\ (x-1)^3 & 1 < x < \infty \end{cases}$ also satisfies IVP

Therefore, solutions exist but are NOT unique

Thm 1: (Existence & Uniqueness)

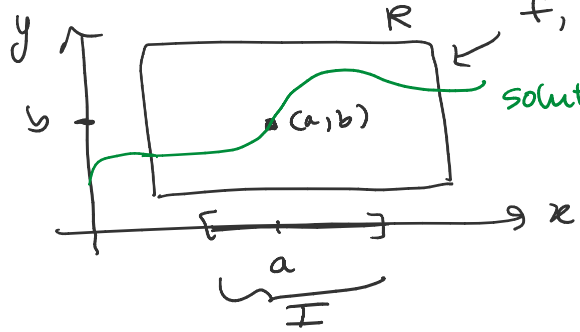
Suppose $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous on some rectangle R in the x-y plane that contains point (a,b). Then for some interval I containing a, the IVP

$$\frac{dy}{dx} = f(x,y) \qquad y(a) = b$$

Some interval I containing a, ...

$$\frac{dy}{dx} = f(x,y) \qquad y(a) = b$$

has a unique solution



$f, \frac{\partial f}{\partial y}$ cts inside R

solution curve through (a,b) exists & is unique

Ex 1: $\frac{dy}{dx} = -3y \qquad y(0) = 5$

$f(x,y) = -3y$ is cts everywhere (for all x, for all y)

$\frac{\partial f}{\partial y} = -3$ is cts everywhere

take $R = (-\infty,\infty) \times (-\infty,\infty)$, $\quad I = (-\infty, \infty)$

$(a,b) = (0,5)$ lies in R

By Thm 1, the IVP has unique solution

(happens to be $y(x) = 5e^{-x}$)

Counter ex 1: $\frac{dy}{dx} = \frac{1}{x} \qquad y(0) = 0$

$f(x,y) = \frac{1}{x}$ is discontinuous at $x = 0$ $\quad (x > 0)$

$\frac{\partial f}{\partial y} = 0$ is continuous everywhere

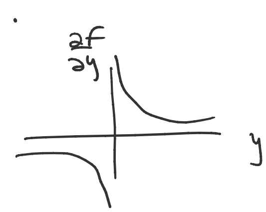
take $R = (0,\infty) \times (-\infty,\infty)$ $\quad$ so $f, \frac{\partial f}{\partial y}$ are cts on R

$(a,b) = (0,0)$ does NOT lie in R

So Thm 1 does not apply

cannot say a unique soln exists.

Counter ex 2: $\frac{dy}{dx} = 3y^{2/3} \qquad y(0) = 0$



$f(x,y) = 3y^{2/3}$ is cts everywhere

$\frac{\partial f}{\partial y} = 3 \cdot \frac{2}{3} y^{-1/3} = \frac{2}{y^{\frac{1}{3}}}$ is discontinuous at $y = 0$

$\frac{\partial f}{\partial y} = 3 \cdot \frac{2}{3} y^{13} = \frac{2}{y^{3}}$ is discontinuous ...

Take $R = (-\infty, \infty) \times (0, \infty)$

$(a,b) = (0,0)$ does not lie in R

Therefore Thm 1 does not apply

can't say a unique soln exists.

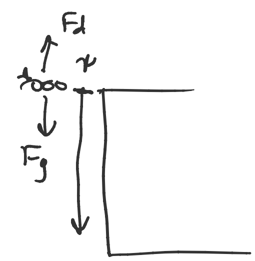
NOTE: If Thm 1 does not apply an IVP, draw a slope field to determine if existence fails or uniqueness fails.

Ex: What happens if an ant is dropped off a bldg?

Forces on ant:

1. gravity $F_g = mg$
   m - mass of ant
   g - grav. accel

2. wind resistence (drag) $F_d = -bv^2$
   v - velocity of ant
   b - drag coeff



Newton's 2nd law:

$$ma = \sum F$$

$$m\frac{dv}{dt} = F_g - F_d = mg - bv^2$$

$$\boxed{\frac{dv}{dt} = g - \frac{b}{m}v^2} \qquad b, m, g > 0$$

1st order, nonlinear ODE

Find equilibrium soln

$$\frac{dv_*}{dt} = 0 = g - \frac{b}{m}v_*^2$$

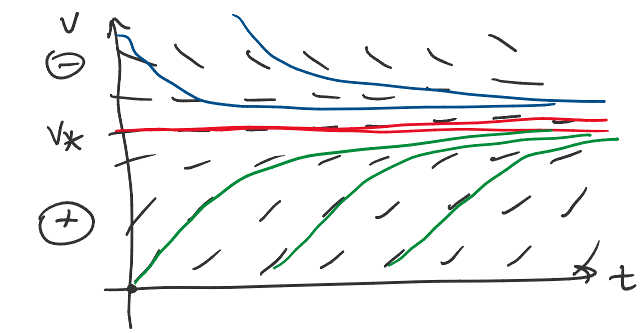
$$\frac{b}{m}v_*^2 = g$$

$$\boxed{v_* = \sqrt{\frac{mg}{b}}}$$

Slope Field

@ v_* slope $= 0$

Slope Field

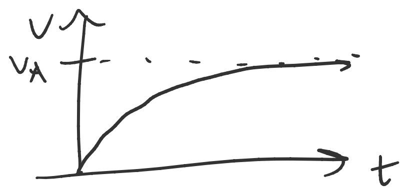


@ v_* slope $= 0$

if $v > v_*$, $\frac{dv}{dt} = g - \frac{b}{m}v^2 < 0$
slope is $\ominus$

if $v < v_*$, $\frac{dv}{dt} = g - \frac{b}{m}v^2 > 0$
slope is $\oplus$

For an ant dropped off a building,
$v(0) = 0$ (no initial velocity)
so solution curve goes through $(0,0)$



$v \to v_*$
as $t \to \infty$

terminal velocity

gravity balances drag

For an ant $\boxed{v_* \sim 2.6 \text{ m/s}}$

So far:

Type	Eqn	Soln Methods
1st order linear (only t on RHS)	$\frac{dy}{dt} = f(t)$	direct integration
2nd order linear (only t on RHS)	$\frac{d^2y}{dt^2} = f(t)$	integrate twice
1st order (non) linear	$\frac{dy}{dt} = f(y,t)$	slope field Existence & Uniqueness

Next time: Separable ODEs.