Section 1.4

☆ Separable Equations & Applications

Warm up: Find the equilibrium solution of

$$\frac{dy}{dx} = x(y-4)$$

eq. soln: $\frac{dy_*}{dx} = 0 = x(y_* - 4) \Rightarrow \boxed{y_* = 4}$

I. Separable ODEs

Def: A 1st order ODE is called <u>separable</u> provided it can be written:

$$\frac{dy}{dx} = g(x)\,h(y)$$

(last time $\frac{dy}{dx} = f(x,y)$)

Ex: $\frac{dy}{dx} = 2xy = (2x)(y)$ separable!

$2x = g(x)$ $h(y) = y$

Solve this by the method of <u>separation</u> <u>of variables</u>

- put all the y-terms on left
  x-terms on right

$$\frac{dy}{y} = 2x\,dx$$

Now "integrate" both sides

$$\int \frac{dy}{y} = \int 2x\,dx$$

$$\ln(y) = \frac{2x^2}{2} + C = x^2 + C$$

exponentiate both sides

$$y(x) = e^{\ln(y)} = e^{x^2 + C} = (e^{x^2})\underbrace{(e^{C})}_{\text{still an unknown const}}$$

rename it A for amplitude

x^2

rename it A
for amplitude

$$\boxed{y(x) = Ae^{x^2}}$$

This is the general solution to the ODE
(b/c it has an undetermined const A)

Later on, we will see that

1st order Linear ODE $\longrightarrow$ Every particular solution comes from the general solution by selecting const c

In constrast, it's common for a nonlinear 1st order ODE to have both

1. general solution (w/ const c)
2. one (or more) particular solns that cannot be obtained by selecting c from the general soln.

$\underbrace{\hspace{6em}}$ called singular solution

Ex: (singular soln)

$$\frac{dy}{dx} = 6x\underbrace{(y-1)^{2/3}}$$

1st order
separable
nonlinear

makes it nonlinear (sec 1.1)

↳ singular solns possible

Find the general soln $\longrightarrow$ separation of variables

$$\frac{dy}{(y-1)^{2/3}} = 6x\,dx$$

$$\int \frac{dy}{(y-1)^{2/3}} = \int 6x\,dx$$

substitution

$\int (y-1)^{2/3} \quad - \quad \int$

u-substitution

$u = y-1$

$du = dy$

$$\int u^{-2/3}\, du = \int 6x\, dx$$

$$3u^{1/3} = \frac{u^{1/3}}{1/3} = \frac{6x^2}{2} + c = 3x^2 + c$$

$$3(y-1)^{1/3} = 3x^2 + c$$

$$(y-1)^{1/3} = x^2 + \frac{c}{3} = x^2 + c_2$$

$$y - 1 = (x^2 + c_2)^3$$

$$\boxed{y(x) = 1 + (x^2 + c)^3}$$ general solution

Look for possible singular solns:

equilibrium solns: $\frac{dy_*}{dx} = 0 = 6x(y_* - 1)^{2/3}$

$y_* - 1 = 0 \qquad \boxed{y_* = 1}$ equilibrium soln

Q: Is $y_* = 1$ a singular solution

$$y(x) = \cancel{1} + (x^2 + c)^3 \overset{?}{=} \cancel{1} = y_*$$

$$(x^2 + c)^3 \overset{?}{=} 0$$

No value of c that makes this true

so $y_* = 1$ does <u>not</u> come from the general soln

so $y_* = 1$ is a <u>singular solution</u>

II. Exponential Function:

Let's consider a special separable ODE

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$$\frac{dy}{dx} = ky \qquad k = \text{constant}$$

1st order, linear, separable $\quad g(x) = k, \; h(y) = y$

solve – separation of variables

$$\frac{dy}{y} = k\,dx$$

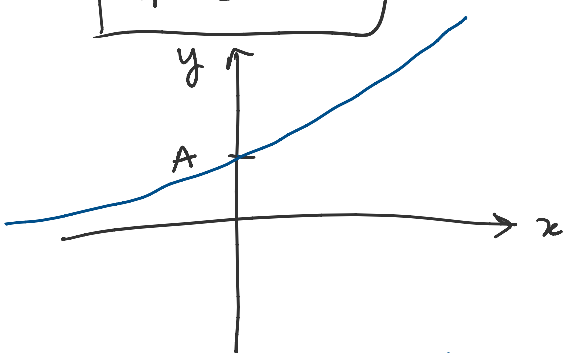
$$\int \frac{dy}{y} = \int k\,dx$$

$$\ln y = kx + C$$

$$y(x) = e^{\ln y} = e^{kx+c} = (e^{kx})(e^{c}) = Ae^{kx}$$

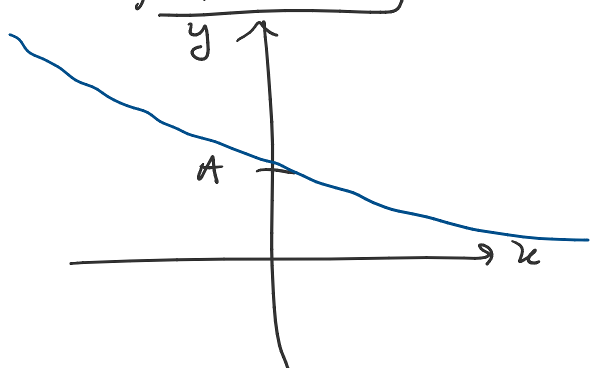
$$\boxed{y(x) = Ae^{kx}}$$ exponential function

if $k > 0$



exponential growth

if $k < 0$



exponential decay

NOTE: The exponential function $y(x) = e^{x}$ is a special function whose derivative is equal to itself: $\frac{dy}{dx} = y$

→ show up in many applications

III. Models of Growth & Decay:

1. Population Growth $\qquad P(t)$ – population

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$$\frac{dP}{dt} = kP$$

$P(t)$ - population
k - net growth rate (# births - # deaths)

2. Compound Interest

$$\frac{dA}{dt} = rA$$

$A(t)$ - \$ in bank acct
r - annual interest rate (compounded continuously)

3. Radioactive Decay

$$\frac{dN}{dt} = -kN$$

$N(t)$ - # of atoms of radioactive isotope
k - decay constant $(k>0)$

ex: for Carbon-14 dating
$k \approx 0.0001216$ (t is measured in years)

4. Drug Elimination

$$\frac{dA}{dt} = -\lambda A$$

$A(t)$ - amount of drug in blood stream
λ - elimination const $(\lambda>0)$

Ex: You work for the FDA and you want to measure the elimination rate of a new drug that's up for approval

$$\frac{dA}{dt} = -\lambda A$$

Find λ

Use separation of variables, we derive

$$A(t) = A_0 e^{-\lambda t}$$ — exp decay

At time $t = 5$ min you measure
$A(5) = 6000$ units of drug in bloodstream

You wait, and at $t = 25$ min, you measure
$A(25) = 200$ units

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<u>Q</u>: Find λ – elimination const

$t = 5$ $\qquad 6000 = A_0 e^{-5\lambda}$

$t = 25$ $\qquad 200 = A_0 e^{-25\lambda}$

divide both equations

$$\frac{\cancel{6000}}{\cancel{200}} = \frac{\cancel{A_0}\, e^{-5\lambda}}{\cancel{A_0}\, e^{-25\lambda}}$$

$$30 = e^{-5\lambda + 25\lambda} = e^{20\lambda}$$

$$\ln(30) = \ln\left(e^{20\lambda}\right) = 20\lambda$$

$$\boxed{\lambda = \frac{\ln(30)}{20} \approx 0.17}$$

<u>Note</u>: we didn't need to solve for A_0 to find λ

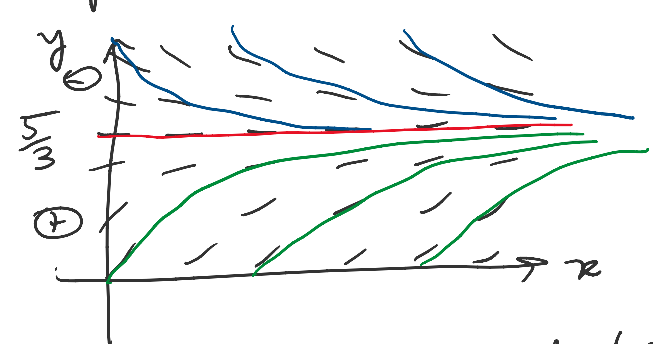
<u>Ex</u>: $\dfrac{dy}{dx} = -3y + 5$ $\qquad$ 1st order linear $\left(\dfrac{dy}{dx} + 3y = 5\right)$ separable

Evaluate solns qualitatively $\longrightarrow$ Slope field

eq. soln: $\dfrac{dy_*}{dx} = 0 = -3y_* + 5$

$3y_* = 5$ $\qquad \boxed{y_* = 5/3}$

Slope Field



@ $y = y_*$ $\quad$ slope $= 0$

if $y > y_*$, $\dfrac{dy}{dx} < 0$ $\quad$ ⊖

if $y < y_*$, $\dfrac{dy}{dx} = -3y + 5 > 0$ $\quad$ ⊕

expect solutions to have a horizontal asymptote at $y_* = 5/3$

$x \to \infty$ $\quad y \to 5/3$

expect solutions ... a horizontal asymptote at $y_* = 5/3$

as $x \to \infty$ $\quad y \to 5/3$

Now solve using sep. of vars.

$$\frac{dy}{dx} = -3y + 5$$

$$\int \frac{dy}{5-3y} = \int dx$$

$\boxed{u = 5-3y}$

$du = -3\,dy$

$$-\frac{1}{3}\int \frac{du}{u} = \int dx$$

$$-\frac{1}{3}\ln u = x + C$$

$$\ln u = -3x - 3C = -3x + C_2$$

$$e^{\ln u} = e^{-3x + C_2}$$

$$u = Ae^{-3x}$$

$$5 - 3y = Ae^{-3x}$$

$$\frac{5 - Ae^{-3x}}{3} = \frac{3y}{3}$$

$$y(x) = \frac{5}{3} - \frac{A}{3}e^{-3x} = \boxed{\frac{5}{3} - Be^{-3x}}$$

general solution

Q: What happens as $x \to \infty$?

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty} \frac{5}{3} - \underbrace{Be^{-3x}}_{\text{exp decay} \to 0} = \frac{5}{3} = y_*$$

solution recovers asymptote at $y_* = 5/3$

... have the initial condition

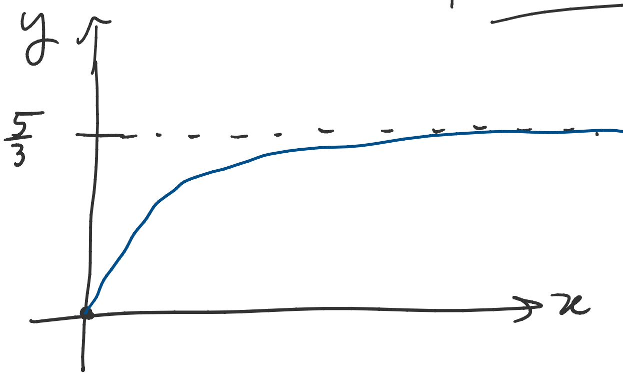
Assume we have the initial condition
$$y(0) = 0$$

Find $B \rightarrow$ $\quad y(0) = 0 = \left[\frac{5}{3} - Be^{-3x} \right] \Big|_{x=0}$

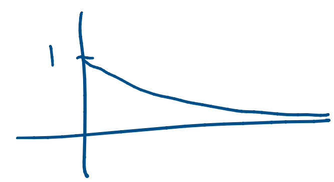
$$0 = \frac{5}{3} - B \rightarrow \boxed{B = \frac{5}{3}}$$

particular solution $\quad y(x) = \frac{5}{3} - \frac{5}{3} e^{-3x}$

$$\boxed{y(x) = \frac{5}{3}\left[1 - e^{-3x}\right]}$$



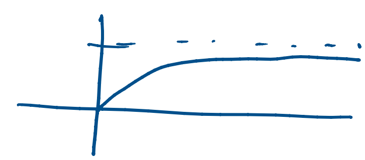
e^{-3x}



$-e^{-3x}$

this matches our solution curves from the slope field.

$1 - e^{-3x}$



So far we have studied:

Type	Eqn	Solution Methods
1st order linear (no y terms on RHS)	$\frac{dy}{dx} = f(x)$	direct integration
2nd order linear (no y terms on RHS)	$\frac{d^2y}{dx^2} = f(x)$	integrate twice
1st order	$\frac{dy}{dx} = f(x, y)$	slope field Existence & Uniqueness

(no y terms

1st order (non) linear	$\frac{dy}{dx} = f(x, y)$	slope field, Existence & Uniqueness
1st order separable	$\frac{dy}{dx} = g(x)h(y)$	separation of variables, Exp growth + decay

Next time: 1st order, linear ODE.