☆ Linear First Order Equations

Warm up: Solve the linear ODE:

$$\frac{dy}{dx} = y + 1$$

using separation of variables.

Q what if RHS was $y + x$?

Ans:

$$\int \frac{dy}{y+1} = \int dx$$

$$e^{\ln(y+1)} = e^{x + C}$$

$$y + 1 = Ae^{x} \rightarrow \boxed{y(x) = -1 + Ae^{x}}$$

I. First Order Linear ODE:

Consider the following ODE

$$\frac{dy}{dx} + a(x)y = q(x)$$

All 1st order linear ODE can be written in the above form.

Ex:

$$x\frac{dy}{dx} + x^2 y = x^3 e^{-x}$$

standard linear form

divide both sides by x

$$\frac{dy}{dx} + xy = x^2 e^{-x}$$

$a(x) = x$

$q(x) = x^2 e^{-x}$

Def: If $a(x) = a$ is a constant, then the ODE is said to have <u>constant coefficients</u>

Def: The function $q(x)$ is called the <u>forcing term</u> or the <u>source term</u>

Consider the IVP

$$(*) \quad \frac{dy}{dx} + ay = q(x) \qquad y(0) = y_0$$

Here $a(x) = a$ is constant $\longrightarrow$ const coeff

1st order, linear

... known methods:

1st order, linear

We can't solve this using known methods:

Rewrite $\frac{dy}{dx} = -ay + q(x)$

- direct ~~integration~~? — No, because y on RHS
- separation ~~of~~ variables? — No, b/c not separable

(could draw a slope field — qualitative sol'ns)

<u>Q</u>: What if we could <u>transform</u> this ODE into something we know how to solve?

$\rightarrow$ change of variables

Define a new fcn: $\mu(x) = e^{ax}$

called an integrating factor

Then let $z(x) = \mu(x)\, y(x) = e^{ax} y(x)$

change of variables.

<u>Q</u>: What ODE does $z(x)$ solve?

$$\frac{dz}{dx} = \frac{d}{dx}\left[e^{ax} y(x)\right] = e^{ax}\frac{dy}{dx} + ae^{ax} y(x)$$

Notice that

$$\frac{dz}{dx} = e^{ax}\Big[\underbrace{\frac{dy}{dx} + a\,y(x)}_{\substack{\text{LHS of ODE} \\ = q(x)}}\Big] = e^{ax} q(x)$$

$$\boxed{\frac{dz}{dx} = e^{ax} q(x)}$$

can solve this!

$\rightarrow$ direct integration

Procedure:

1. Find integrating factor $\mu(x) = e^{ax}$
2. Change of variables $z(x) = e^{ax} y(x)$
3. New ODE: $\frac{dz}{dx} = e^{ax} q(x)$
4. Solve by direct integration
   $z(x) = \int \frac{dz}{dx} = \int e^{ax} q(x)\, dx$
5. Change back to $y(x)$
   $y(x) = e^{-ax} z(x) = e^{-ax} \int e^{ax} q(x)\, dx$

Ex: IVP: $\underline{\frac{dy}{dx} + 5y} = e^{3x}$ $\quad y(0) = 2$

Here $a = 5$, $q(x) = e^{3x}$

1. Find the integrating factor: $\mu(x) = e^{ax} = e^{5x}$
2. Change of variables: $z(x) = e^{5x} y(x)$
3. Derive new ODE

$$\frac{dz}{dx} = \frac{d}{dx}\left[e^{5x} y(x)\right] = e^{5x} y' + 5e^{5x} y$$
$$= e^{5x}\left[\underline{y' + 5y}\right] = e^{5x}\left(e^{3x}\right) = e^{8x}$$

$$\frac{dz}{dx} = e^{8x}$$

4. Solve the ODE by direct integration

$$z(x) = \int \frac{dz}{dx} = \int e^{8x}\, dx = \frac{e^{8x}}{8} + C$$

5. Change back to $y(x)$

$z(x) = e^{5x} y(x)$ $\qquad y(x) = e^{-5x} z(x)$

$$y(x) = e^{-5x}\left[\frac{e^{8x}}{8} + C\right] = \frac{1}{8} e^{3x} + C e^{-5x}$$

6. Find the initial condition

$$y(0) = 2 = \left[\frac{1}{8} e^{3x} + C e^{-5x}\right]\Big|_{x=0}$$

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$$2 = \left[\tfrac{1}{8} e^{0} + C e^{0}\right] = \tfrac{1}{8} + C$$

$$C = 2 - \tfrac{1}{8} = \tfrac{16-1}{8} = \tfrac{15}{8}$$

particular soln : $\boxed{y(x) = \tfrac{1}{8} e^{3x} + \tfrac{15}{8} e^{-5x}}$

II. Integrating Factor :

Q: Why do we make the change of variables to
$z(x) = e^{ax} y(x)$

Q: What if $a(x)$ is NOT a constant? What change of variables does we make?

Consider the ODE

$$(\square) \qquad \frac{dy}{dx} + a(x)\, y = q(x)$$

variable coefficients

Want to make a change of variables, so let's multiply both sides by an unknown function $\mu(x)$ — called integrating factor

$$\underbrace{\mu(x)\, y'(x) + a(x)\mu(x)\, y(x)} = \mu(x)\, q(x)$$

Key step: Observe that LHS almost looks like the product rule for derivatives.

product rule $\quad \frac{d}{dx}\left[f(x)\, y(x)\right] = f(x)\, y'(x) + f'(x)\, y(x)$

LHS $\quad = \mu(x)\, y'(x) + a(x)\mu(x)\, y(x)$

To set these equal, we need
$f(x) = \mu(x)$ and $f'(x) = a(x)\mu(x)$

putting these two together

$\mu'(x) = a(x)\,\mu(x)$

Putting these two together

$$\mu'(x) = a(x)\mu(x)$$

$$\frac{d\mu}{dx} = a(x)\mu \qquad \text{separable! sep of variables}$$

$$\int \frac{d\mu}{\mu} = \int a(x)\,dx$$

$$e^{\ln \mu} = e^{\int a(x)dx}$$

Check: when $a(x)=a$ then $\mu(x) = \exp\left[\int a\,dx\right] = \exp\left[ax\right] = e^{ax}$ ✓

$$\boxed{\mu(x) = \exp\left[\int a(x)\,dx\right]}$$

integrating factor

We have our integrating factor!
Go back to ODE

$$\mu(x)y' + \underbrace{a(x)\mu(x)}_{\text{by Def of } \mu(x)}y = \mu(x)q(x)$$

$a(x)\mu(x) = \mu'(x)$

product rule

$$\mu(x)y' + \mu'(x)y = \mu(x)q(x)$$

$$\frac{d}{dx}\left[\mu(x)y(x)\right] = \mu(x)q(x)$$

Def
$z(x) = \mu(x)y(x)$
$z' = \mu(x)q(x)$

integrate both sides wrt x

$$\mu(x)y(x) = \int \frac{d}{dx}\left[\mu(x)y(x)\right] = \int \mu(x)q(x)\,dx$$

Solve for $y(x)$

$$\boxed{y(x) = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx}$$

We say that we solve this ODE by <u>using</u> <u>the</u> <u>integrating</u> <u>factor</u>

<u>Def</u>: The <u>integrating</u> <u>factor</u> for $(\square)$ is

Def: The <u>integrating factor</u> for (□) is

$$\mu(x) = \exp\left(\int a(x)\,dx\right)$$

Ex: Find the integrating factor for the following ODE:

(a) $y' + y = x$ $\quad a=1 \quad \mu(x) = \exp\left(\int 1\,dx\right) = \boxed{e^{x}}$

(b) $y' - 10y = e^{3x}$ $\quad a=-10 \quad \mu(x) = \exp\left(\int -10\,dx\right) = \boxed{e^{-10x}}$

(c) $y' - (\sin x)\,y = \sin x$ $\quad a(x) = -\sin x$

$$\mu(x) = \exp\left(\int -\sin x\,dx\right) = \exp(\cos(x)) = \boxed{e^{\cos(x)}}$$

Ex: Solve the IVP using the integrating factor

$$\frac{dy}{dx} - \sin(x)\,y = \sin(x) \qquad y(0) = 2$$

1. Find the integrating factor $\mu(x) = e^{\cos(x)}$

2. Multiply both sides of the ODE by $\mu(x)$ and simplify

$$e^{\cos(x)} y' - \sin(x) e^{\cos(x)} y = \sin(x) e^{\cos(x)}$$

pull out a derivative on LHS

$$\frac{d}{dx}\left[e^{\cos(x)} y(x)\right] = \sin(x) e^{\cos(x)}$$

3. Solve the new ODE – integrate both sides

$$\int \frac{d}{dx}\left[e^{\cos(x)} y(x)\right] = \int \sin(x) e^{\cos(x)}\,dx$$

$u = \cos(x) \quad du = -\sin(x)\,dx$

$$e^{\cos(x)} y(x) = \int -e^{u}\,du = -e^{u} + C$$

$$e^{\cos(x)} y(x) = -e^{\cos(x)} + C$$

Solve for $y(x)$

$$y(x) = e^{-\cos(x)}\left[-e^{\cos(x)} + C\right]$$

$$y(x) = -1 + Ce^{-\cos(x)} \quad \text{general solution}$$

4. Find the initial condition

$$y(0) = 2 = \left[-1 + Ce^{-\cos(x)}\right]\Big|_{x=0}$$

$$2 = -1 + Ce^{-\cos(0)} = -1 + Ce^{-1}$$

$$3 = \frac{C}{e} \Rightarrow C = 3e$$

particular soln

$$y(x) = -1 + (3e)e^{-\cos(x)}$$

$$\boxed{y(x) = -1 + 3e^{-\cos(x)+1}}$$

III. Comparison:

We've seen this kind of procedure before

→ similar to completing the square

integrating factor

eqn: $c_1(x)\frac{dy}{dx} + c_0(x)y = g(x)$

rearrange $\frac{dy}{dx} + a(x)y = q(x)$

multiply by $\mu(x) = \exp\left(\int a(x)dx\right)$

$$\underbrace{\mu(x)y' + a(x)\mu(x)y}_{} = \mu(x)g(x)$$

$$\frac{d}{dx}\left[\mu(x)y(x)\right] = \mu(x)g(x)$$

Solve for $y(x)$

$$\mu(x)y(x) = \int \mu(x)g(x)dx$$

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)g(x)dx$$

completing the square

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x = \frac{-c}{a}$$

add $+\frac{b^2}{4a^2}$

$$\underbrace{x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}}_{} = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2-4ac}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

solving for x

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

Next: – more integrating factor
– applications