Linear First Order Equations:

Warm up: Find the integrating factor for:

$$\frac{dy}{dx} + \frac{3}{x}y = x - 4$$

$a(x) = \frac{3}{x}$ $\quad \mu(x) = \exp\left(\int a(x)dx\right) = \exp\left(\int \frac{3}{x}dx\right)$

$$= \exp\left(3\ln(x)\right) = \exp\left(\ln(x^3)\right)$$

$$\boxed{\mu(x) = x^3}$$

I. First order linear ODEs:

Last lecture:

$$\frac{dy}{dx} + a(x)y = q(x), \quad y(x_0) = y_0$$

solved this using the integrating factor

procedure:

1. Find the integrating factor:
$$\mu(x) = \exp\left(\int a(x)dx\right)$$

2. Multiply both sides of ODE by $\mu(x)$ and factor out the derivative
$$\mu(x)\frac{dy}{dx} + a(x)\mu(x)y = \mu(x)q(x)$$
Factor out derivative
(Chose $\mu' = a(x)\mu(x)$)
$$\frac{d}{dx}\left[\mu(x)y\right] = \mu(x)q(x)$$

3. Integratie both sides (wrt x) and solve for $y(x)$
4. Plug in the initial condition $y(x_0) = y_0$

Ex: $\frac{dy}{dx} + \frac{3}{x}y = x - 4, \quad y(1) = -2$

1. Find the integrating factor

$$\mu(x) = \exp\left(\int \frac{3}{x}dx\right) = x^3$$

1. Find the integrating factor

$$\mu(x) = \exp\left(\int \frac{3}{x}\,dx\right) = x^3$$

2. Multiply both sides by $\mu(x)$

$$x^3 \frac{dy}{dx} + \frac{3}{x} x^3 y = x^3 (x-4)$$

$$x^3 \frac{dy}{dx} + 3x^2 y = x^4 - 4x^3$$

$$\frac{d}{dx}\left[x^3 y\right] = x^4 - 4x^3$$

3. Integrate both sides

$$\int \frac{d}{dx}\left[x^3 y\right] = \int x^4 - 4x^3 \, dx$$

$$x^3 y = \frac{x^5}{5} - \frac{\cancel{4}x^4}{\cancel{4}} + C$$

$$y(x) = x^{-3}\left[\frac{x^5}{5} - x^4 + c\right]$$

$$y(x) = \frac{x^2}{5} - x + Cx^{-3}$$

4. plug in initial condition

$$y(1) = -2 = \left[\frac{x^2}{5} - x + Cx^{-3}\right]\Big|_{x=1}$$

$$-2 = \frac{(1)^2}{5} - 1 + C(1)^{-3} = \frac{1}{5} - 1 + C$$

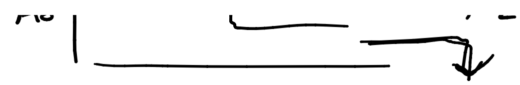
$$-1 = \frac{1}{5} + C \qquad C = -1 - \frac{1}{5} = \frac{-6}{5}$$

$$\boxed{y(x) = \frac{1}{5}x^2 - x - \frac{6}{5}x^{-3}}$$ particular solution

II. Mixing Problems:

Tank that holds a solution — mixture of solute and solvent

Track 2 functions:



Track 2 functions:

$V(t)$ - volume of solution in tank

$A(t)$ - amount of solute in tank

Define $c(t) = \dfrac{A(t)}{V(t)}$ — concentration

Goal: Derive 2 IVPs for $V(t)$, and $A(t)$

constant

Constants:

V_0 — initial volume (gal)

c_1 — concentration of solution coming in (lb/gal)

A_0 — initial amount (lb)

r_1 — rate coming in (gal/s)

r_2 — rate going out (gal/s)

c_2 - concentration out (lb/gal)

Assume that solution coming out is well mixed

$c_2 = c_2(t) = \dfrac{A(t)}{V(t)}$ function of t

Q: How does the volume change? $V(t)$

in time Δt

volume $\Delta V = \underset{\text{volume in}}{V_1} - \underset{\text{volume out}}{V_2}$ (gal)

$= r_1 \Delta t - r_2 \Delta t$ (gal)

$$\frac{dV}{dt} \approx \frac{\Delta V}{\Delta t} = \frac{r_1 \Delta t - r_2 \Delta t}{\Delta t} = r_1 - r_2$$

gives rise to an IVP

$$\begin{cases} \dfrac{dV}{dt} = r_1 - r_2 \\ V(0) = V_0 \end{cases}$$

Q: How does the amount of solution change?

Look ΔA in time Δt

Recall: $c = \dfrac{A}{V} \Rightarrow A = cV$

so, in Δt, $\Delta A = A_1 - A_2$

$= c_1 V_1 - c_2 V_2$

$= c_1 r_1 \Delta t - c_2 r_2 \Delta t$

Constants

$$= c_1 v_1 - c_2 v_2$$
$$= c_1 r_1 \Delta t - c_2 r_2 \Delta t$$

So $\dfrac{dA}{dt} \approx \dfrac{\Delta A}{\Delta t} = \dfrac{c_1 r_1 \Delta t - c_2 r_2 \Delta t}{\Delta t} = c_1 r_1 - c_2 r_2$

Constants

is a fcn of time

Recall: $c_2 = c_2(t) = \dfrac{A(t)}{V(t)}$

$$\begin{cases} \dfrac{dA}{dt} = c_1 r_1 - \dfrac{A(t)\, r_2}{V(t)} \\ A(0) = A_0 \end{cases}$$

To solve a mixing problem, need to solve

$$\begin{cases} \dfrac{dV}{dt} = r_1 - r_2 \\ V(0) = V_0 \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{dA}{dt} = c_1 r_1 - \dfrac{A(t)\, r_2}{V(t)} \\ A(0) = A_0 \end{cases}$$

Ex: A tank contains 4L of water, in which is dissolved 32 g of some chemical.
A solution containing 2g/L of chemical flows into the tank at rate 4L/min
The well-stirred mixture flows out at rate 2L/min

Q: a) Determine the amount of chemical in tank at t = 10min $A(t=10)$

b) Find the concentration of the solution in tank after 10 min $C(t=10)$

$A_0 = 32g$

$c_1 = 2g/L$

$c_2 = \dfrac{A(t)}{V(t)}$

$V_0 = 4L$

$r_1 = 4\ L/min$

$r_2 = 2L/min$

Write down equations:

$$\frac{dV}{dt} = r_1 - r_2 = 4 - 2 = 2$$

$$\begin{cases} \dfrac{dV}{dt} = 2 \end{cases}$$

$$\frac{dV}{dt} = r_1 - r_2 = 4 - 2 = 2 \qquad \left\{ \begin{array}{l} \frac{dV}{dt} = 2 \\ V(0) = 4 \end{array} \right.$$

$$V(0) = V_0 = 4$$

$$\left\{ \begin{array}{l} \frac{dA}{dt} = r_1 c_1 - \frac{r_2 A(t)}{V(t)} = (4)(2) - \frac{(2)A}{V(t)} \leftarrow \text{Need to solve for } V(t) \text{ first} \\ A(0) = A_0 = 32 \end{array} \right.$$

<u>Volume</u>: $\frac{dV}{dt} = 2 \qquad V(0) = 4$ 1st order linear

solve by direct integration

$$\int \frac{dV}{dt} = \int 2\,dt \qquad V(0) = 4 = [2t + c]\Big|_{t=0}$$

$$V(t) = 2t + C \qquad 4 = C$$

$$\boxed{V(t) = 2t + 4}$$

<u>Amount</u>: $\left\{ \begin{array}{l} \frac{dA}{dt} = 8 - \frac{2A}{2t+4} = 8 - \frac{A}{t+2} \\ A(0) = 32 \end{array} \right.$

Rewrite ODE: 1st order Linear

$$\frac{dA}{dt} + \left(\frac{1}{t+2}\right) A = 8, \quad A(0) = 32$$

solve this using the integrating factor

$$a(t) = \frac{1}{t+2}$$

1. Find $\mu(t)$

I.F. $\mu(t) = \exp\left(\int \frac{1}{t+2}\,dt\right) = \exp\left(\ln(t+2)\right)$

$$\mu(t) = t + 2$$

2. Multiply both sides by $\mu(t)$

$$(t+2)\frac{dA}{dt} + \cancel{(t+2)}\left(\frac{1}{\cancel{t+2}}\right) A = 8(t+2)$$

$$\frac{d}{dt}\left[(t+2)A\right] = 8t + 16$$

Integrate both sides

$\frac{d}{dt}[\,(t+2)A\,]$

3. Integrate both sides

$$\int \frac{d}{dt}\left[(t+2)A\right] = \int 8t+16 \;dt$$

$$(t+2)A = \frac{8t^2}{2} + 16t + C$$

$$A(t) = \frac{4t^2+16t+C}{t+2}$$

4. Solve for initial condition

$$A(0) = 32 = \left[\frac{4t^2+16t+C}{t+2}\right]\Big|_{t=0} = \frac{C}{2}$$

$$C = 2\cdot 32 = 64$$

so $\boxed{A(t) = \frac{4t^2+16t+64}{t+2}}$

(a) What is A @ $t = 10\,\text{min}$?

$$A(t=10) = \left[\frac{4t^2+16t+64}{t+2}\right]\Big|_{t=10} = \frac{4(100)+16(10)+64}{10+2}$$

$$= \frac{624}{12} = \boxed{52\,g}$$

(b) What is C @ $t = 10\,\text{min}$?

$C(t) = \frac{A(t)}{V(t)}$ at $t = 10\,\text{min}$

$$C(10) = \frac{A(10)}{V(10)} = \frac{52}{[2t+4]\big|_{t=10}} = \frac{52}{24} = \boxed{\frac{13}{6}\,g/L}$$

Ex: What if we change the previous example so that $r_2 = 4\,L/\text{min}$

(a) Find $A(t)$

(b) Find $\lim_{t\to\infty} A(t)$

(c) Find the time t_* when $A(t_*) = \frac{1}{2}A_0 = 16g$

... $t \to \infty$

(c) Find the time $t*$ when $A(t*) = \frac{1}{2} A_0 = 16g$

parameters: $V_0 = 4L$ $\quad A_0 = 32g$

$r_1 = 4L/min$ $\quad c_1 = 2g/L$

$r_2 = 4L/min$ $\quad c_2 = A(t)/V(t)$

Volume: $\frac{dV}{dt} = r_1 - r_2 = 4 - 4 = 0$ $\quad \begin{cases} \frac{dV}{dt} = 0 \\ V(0) = 4 \end{cases}$

$V(0) = 4$

solve by direct integration

$\int \frac{dV}{dt} = \int 0 \, dt$ $\quad V(0) = 4 = C$

$V(t) = C$ $\quad$ so $\boxed{V(t) = 4}$ volume doesn't change

Amount: $\frac{dA}{dt} = r_1 c_1 - \frac{r_2 A}{V(t)} = (4)(2) - \frac{\cancel{4} A}{\cancel{4}} = 8 - A$

$A(0) = A_0 = 32$

$\begin{cases} \frac{dA}{dt} = 8 - A \\ A(0) = 32 \end{cases}$ $\quad$ 1st order linear $>$ integrating factor

separable – could solve using separation of variables

solve using the integrating factor

Rewrite: $\frac{dA}{dt} + A = 8$, $\quad A(0) = 32$

1. Find $\mu(t) = \exp\left(\int 1 \, dt\right) = \exp(t) = e^t$

2. Multiply both sides by $\mu(t)$

$$e^t \frac{dA}{dt} + e^t A = 8e^t$$

factor out the deriv.

$$\frac{d}{dt}\left[e^t A\right] = 8e^t$$

3. Integrate both sides

at

3. Integrate both sides

$$\int \frac{d}{dt}\left[e^{t}A\right] = \int 8e^{t}\,dt$$

$$e^{t}A = 8e^{t} + C$$

$$A = e^{-t}\left[8e^{t} + C\right]$$

$$A(t) = 8 + Ce^{-t}$$

4. Plug in the initial condition

$$A(0) = 32 = \left[8 + Ce^{-t}\right]\Big|_{t=0}$$

$$32 = 8 + C\cancel{e^{-0}} = 8 + C \rightarrow C = 24$$

(a) Find $\boxed{A(t) = 8 + 24e^{-t}}$

(b) $\lim_{t\to\infty} A(t) = \lim_{t\to\infty} 8 + 24e^{-t} = \boxed{8}$

exp decay as $t\to\infty$ $\rightarrow 0$

(c) At what t_* does $A(t_*) = 16$?

solve for t_*

$$A(t_*) = 16 = 8 + 24e^{-t_*}$$

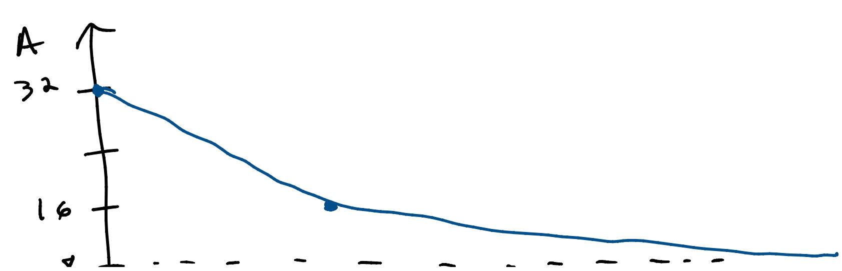
$$8 = 24e^{-t_*}$$

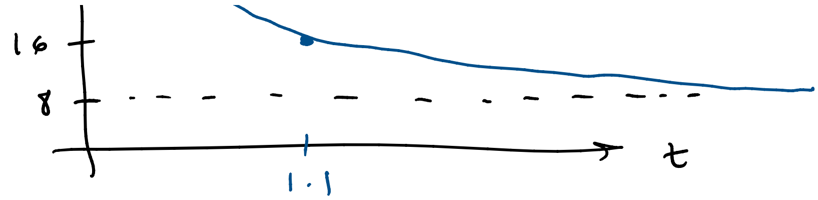
$$\frac{1}{3} = \frac{8}{24} = e^{-t_*}$$

$$\ln\left(\tfrac{1}{3}\right) = \ln\left(e^{-t_*}\right) = -t_*$$

so $t_* = -\ln\left(\tfrac{1}{3}\right) \approx \boxed{1.1 \text{ min}}$

(d) Plot $A(t)$





16
8
t
1.1