✱ Substitution Methods & Exact Equations

Warm up: Solve the separable ODE: $\frac{dy}{dx} = \frac{x}{y^2}$

Ans: $\int y^2 \, dy = \int x \, dx \rightarrow \frac{y^3}{3} = \frac{x^2}{2} + C \rightarrow y^3 = \frac{3}{2}x^2 + C$

$$\boxed{y(x) = \left(\frac{3}{2}x^2 + C\right)^{1/3}}$$

I. Substitution Methods:

Last 2 classes → solve 1st order linear eqns
→ integrating factor → solve all 1st order linear eqns

In general, a 1st order, nonlinear ODE might not be solvable
→ but in case nonlinear is solveable here are some methods that work

GOAL: Transform the ODE from one form to another (that we can solve)

Ex: $\frac{dy}{dx} = (x+y+3)^2$ — 1st order, nonlinear, NOT separable

Q: Can we transform this into a separable ODE?

Change of variables: $v = (x+y+3)$

Find an ODE for v

$$\frac{dv}{dx} = \frac{d}{dx}(x+y+3) = 1 + \frac{dy}{dx} + 0 = 1 + (x+y+3)^2$$

$$\boxed{\frac{dv}{dx} = 1 + v^2}$$

1st order, non linear, separable

Solve using separation of variables

$$\frac{dv}{1+v^2} = dx$$

$$\frac{dv}{1+v^2} = dx$$

$$\int \frac{dv}{1+v^2} = \int dx$$

$$\tan^{-1}(v) = x + C$$

$$v = \tan(x+c)$$

convert v back into x and y

$$x+y+3 = v = \tan(x+c)$$

$$\boxed{y(x) = \tan(x+C) - x - 3}$$

NOTE: The choice of v is not always obvious
Today — Look at 2 common substitutions

II. Homogeneous ODE:

Def: A 1st order ODE is homogeneous if it can be written in the form: $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$

Ex:

(a) $\frac{dy}{dx} = \frac{2y^3 - 5x^2 y}{x^3} = \frac{2y^3}{x^3} - \frac{5x^2 y}{x^3} = 2\left(\frac{y}{x}\right)^3 - 5\left(\frac{y}{x}\right)$ ✓ homogeneous

(b) $\frac{dy}{dx} = \frac{2y^3 - 5xy}{x^3} = \frac{2y^3}{x^3} - \frac{5xy}{x^3} = 2\left(\frac{y}{x}\right)^3 - \frac{5}{x}\left(\frac{y}{x}\right)$ ✗ NOT homogeneous

(c) $\frac{dy}{dx} = \frac{x-y}{x+y} = \frac{(x-y)\frac{1}{x}}{(x+y)\frac{1}{x}} = \frac{1 - y/x}{1 + y/x}$ ✓ homogeneous

(d) $\frac{dy}{dx} = \ln(y) + \ln(x) = \ln(yx)$ ✗ homogeneous

(e) $\frac{dy}{dx} = \ln(y) - \ln(x) = \ln\left(\frac{y}{x}\right)$ ✓ homogeneous

Method for solving homogeneous ODE:

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \longrightarrow \text{change of variables}$$

1. Change of variables let $v = y/x$
2. Rewrite ODE in terms of v and x and dv/dx
3. Solve the ODE in v
4. convert v back to y and x

Ex: $$\frac{dy}{dx} = \cos^2\left(\frac{y}{x}\right) + \frac{y}{x}$$ ✓ homogeneous

1. Change of variables: $v = \frac{y}{x}$

2. Rewrite ODE in terms of v and x

$$\frac{dy}{dx} = \cos^2(v) + v$$

Need to find $\frac{dy}{dx}$ in terms of v and x

Recall $v = \frac{y}{x} \iff xv = y$

$$\frac{d}{dx}(xv) = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{d}{dx}(xv(x)) = (1)v + x\frac{dv}{dx} = v + x\frac{dv}{dx}$$

$$\cancel{v} + x\frac{dv}{dx} = \frac{dy}{dx} = \cos^2(v) + \cancel{v}$$

$$x\frac{dv}{dx} = \cos^2(v)$$

$$\boxed{\frac{dv}{dx} = \frac{\cos^2(v)}{x}}$$

3. Solve for v — 1st order, nonlinear, separable

separation of variables

$$\frac{dv}{\cos^2(v)} = \frac{dx}{x}$$

$$\int \frac{dv}{\cos^2(v)} = \int \frac{dx}{x}$$

$$\tan(v) = \ln(x) + c$$

$$v = \tan^{-1}(\ln(x) + c)$$

4. Convert v back to y and x

Recall $v = \frac{y}{x} = \tan^{-1}(\ln|x| + c)$

$$\boxed{y(x) = x \tan^{-1}(\ln|x| + c)}$$

III. Bernoulli Eqns:

Def: A 1st order ODE of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \quad (*)$$

is called a Bernoulli equation

NOTE: if $n = 0$ $\quad \frac{dy}{dx} + P(x)y = Q(x)$ linear

if $n = 1$ $\quad \frac{dy}{dx} + P(x)y = Q(x)y$

$$\frac{dy}{dx} + [P(x) - Q(x)]y = 0 \quad \text{linear}$$

any other n $\longrightarrow$ this is nonlinear

HW: (Sec 1.6 # 56) use the substitution $v = y^{1-n}$ to transform $(*)$ into:

$$\frac{dv}{dx} + (1-x)P(x)v = (1-n)Q(x)$$

Now a 1st order linear equation
↳ solve this using the integrating factor.

Method to Solve Bernoulli:

1. Change of variables: $v = y^{1-n}$

ODE in terms of v and x

1. Change of variables: $v = y$
2. Rewrite the ODE in terms of v and x
3. Solve for $v(x)$
4. Transform v back to y and x

Ex: $2xy \frac{dy}{dx} = 4x^2 + 3y^2$ Not in Bernoulli form

0. Rewrite ODE in Bernoulli form
Divide both sides by $2xy$

$$\frac{dy}{dx} = \frac{4x^2+3y^2}{2xy} = \frac{4x^{\cancel{2}}}{\cancel{2x}y} + \frac{3y^{\cancel{2}}}{2x\cancel{y}}$$

$$\frac{dy}{dx} = (2x)y^{-1} + \left(\frac{3}{2x}\right)y$$

$$\frac{dy}{dx} + \left(\frac{-3}{2x}\right)y = (2x)y^{-1}$$

Bernoulli form: $\frac{dy}{dx} + P(x)y = Q(x)y^n$

$P(x) = \frac{-3}{2x}$
$Q(x) = 2x$
$\boxed{n = -1}$

1. Change of variables $v = y^{1-n} = y^{1-(-1)} = y^2$

$$v = y^2 \iff y = v^{1/2}$$

2. Rewrite ODE in terms of v and x

$$\frac{dy}{dx} + \left(-\frac{3}{2x}\right)y = (2x)y^{-1}$$

plug these all in

$y = v^{1/2}$ $\quad y^{-1} = (v^{1/2})^{-1} = v^{-1/2}$

$\frac{dy}{dx} = \frac{d}{dx}(v^{1/2}) = \frac{1}{2}v^{-1/2}\frac{dv}{dx}$

$$\left(\frac{1}{2}v^{-1/2}\frac{dv}{dx}\right) + \left(\frac{-3}{2x}\right)v^{1/2} = (2x)v^{-1/2}$$

multiply both sides by $2v^{1/2}$

$\left(\quad dx \right) \quad (2x)$

multiply both sides by $2v^{1/2}$

$$(\cancel{2v^{1/2}})\left(\cancel{\tfrac{1}{2}v^{-1/2}}\right)\frac{dv}{dx} + (\cancel{2}v^{1/2})\left(\frac{-3}{\cancel{2}x}\right)v^{1/2} = 2\cancel{v^{1/2}}(2x)\cancel{v^{-1/2}}$$

$$\boxed{\frac{dv}{dx} + \left(-\frac{3}{x}\right)v = 4x}$$ ODE in v

3. Solve the ODE for v

1st order, linear

Solve it using the integrating factor!

I.F. $\mu(x) = \exp\left(\int -\frac{3}{x}\,dx\right) = \exp(-3\ln(x))$

$$= \exp(\ln(x^{-3})) = \boxed{x^{-3} = \mu(x)}$$

Multiply both sides by $\mu(x)$

$$x^{-3}\frac{dv}{dx} + \left(-\frac{3}{x}\right)x^{-3}v = 4x(x^{-3})$$

$$x^{-3}\frac{dv}{dx} - 3x^{-4}v = 4x^{-2}$$

$$\int \frac{d}{dx}\left[x^{-3}v\right] = \int 4x^{-2}$$

$$x^{-3}v = \frac{4x^{-1}}{-1} + C$$

$$v = x^3\left(-\frac{4}{x} + C\right)$$

$$v = -4x^2 + Cx^3$$

4. Convert v back to y and x

Recall $v = y^2$

$$y^2 = v = -4x + Cx^3$$

$$\boxed{y(x) = \pm(-4x + Cx^3)^{1/2}}$$

$$\boxed{y(x) = \pm(-4x + Cx)}$$

Bernoulli : 1. change of variables $v = y^{1-n}$
2. ODE for v — is 1st order linear
3. solve using the integrating factor.
4. convert back to y and x

Summary:

Goal: Substitute $v = v(x)$ into ODE to transform it from nonlinear into something we know how to solve

1st order Nonlinear $\xrightarrow[v]{}$
1. separable
2. 1st order linear
3. $\frac{dv}{dx} = f(x)$ direct integration

Type	Eqn	Method
homogeneous	$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$	substitution $v = y/x$
Bernoulli	$\frac{dy}{dx} + P(x)y = Q(x)y^n$	Substitution $v = y^{1-n}$

Next class : finish 1.6 — Exact eqns