Section 1.6 - Part 2

*Substitution Methods & Exact Equations:

Warm up: Fill in the chart below:

Type	Eqn	Method
homogeneous	$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$	substitution $v = \frac{y}{x}$
Bernoulli	$\frac{dy}{dx} + P(x)y = Q(x)y^n$	substitution $v = y^{1-n}$

I. Exact Equations:

A general solution of a 1st order ODE can be written in implicit form:

$$F(x, y(x)) = C$$

where c is const

Ex: $xy^2 - \frac{2x}{\ln(x)} = C$ general soln

We can recover the ODE by taking a derivative of both sides:

$$\frac{d}{dx}\left[F(x, y(x)) = C\right]$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

we can rewrite as:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

1st order ODE

Sometimes we write

$$M(x,y)\,dx + N(x,y)\,dy = 0 \qquad (*)$$

called the differential form

a function $F(x,y)$ such that

Def: If there exists a function $F(x,y)$ such that
$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N$$
then the ODE $(*)$ is called ***exact*** and the implicit equation
$$F(x,y) = C \quad \text{solves } (*)$$

Note: There is a quick check for exactness

1. Take the y-derivative of M
$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial^2 F}{\partial y \partial x}$$
2. Take the x-derivative of N
$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial^2 F}{\partial x \partial y}$$
3. Assume that $F(x,y)$ is "nice" then
$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$$

That implies
$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

An ODE like $(*)$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Ex: Is the following ODE exact?

(a) $$y^3 dx + 3xy^2 dy = 0$$

$M = y^3$ $\qquad$ $N = 3xy^2$

$$\frac{\partial M}{\partial y} = 3y^2 = \frac{\partial N}{\partial x} = 3y^2$$

so yes, *exact*

(b) $y\,dx + 3x\,dy = 0$

$M = y$ $\qquad N = 3x$

$\frac{\partial M}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = 3$ NOT exact.

Method for Solving an Exact ODE $(*)\ M(x,y)dx + N(x,y)dy = 0$

1. Check that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (if not, stop)
2. Integrate M wrt x (since $M = \frac{\partial F}{\partial x}$)

   $F_M = \int M(x,y)\,dx$
3. Integrate N wrt y (since $N = \frac{\partial F}{\partial y}$)

   $F_N = \int N(x,y)\,dy$
4. Combine F_M and F_N, being sure not to duplicate terms.

Ex: $(6xy - y^3)dx + (4y + 3x^2 - 3xy^2)dy = 0$

$M = 6xy - y^3$ $\qquad N = 4y + 3x^2 - 3xy^2$

1. Check if ODE exact

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[6xy - y^3\right] = 6x - 3y^2$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[4y + 3x^2 - 3xy^2\right] = 0 + 6x - 3y^2$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 6x - 3y^2 \rightarrow \text{exact.}$$

2. Integrate M wrt x

$$F_M = \int (6xy - y^3)\,dx = \frac{6x^2}{2}y - xy^3 + c_1(y)$$

$$F_M = 3x^2y - xy + c_1(y)$$

$c_1(y)$ since

$F_M = 3x\ y\ -y$

Note: add on a fcn $c_1(y)$ since $M = \frac{\partial F}{\partial x}$ will not capture any information about purely y-terms

3. Integrate N wrt y

$$F_N = \int 4y + 3x^2 - 3xy^2 \, dy$$

$$= \frac{4y^2}{2} + 3x^2y - \frac{3xy^3}{3} + C_2(x)$$

$$F_N = 2y^2 + 3x^2y - xy^3 + C_2(x)$$

4. Combine F_M and F_N

$$F_M = \underline{3x^2y} - \underline{xy^3} + \underline{c_1(y)}$$

$$F_N = \underline{3x^2y} - \underline{xy^3} + \underline{2y^2} + C_2(x)$$

common terms : $3x^2y - xy^3$

We see that : $c_1(y) = 2y^2$

$c_2(x) = 0$

so $F(x,y) = 3x^2y - xy^3 + 2y^2$

The solution to the ODE

$$\boxed{3x^2y - xy^3 + 2y^2 = C}$$

Ex: $\frac{1}{x}\,dx + 6y^2\,dy = 0$

1. Check if exact:

$M = \frac{1}{x}$ $\qquad \frac{\partial M}{\partial y} = 0$

$N = 6y^2$ $\qquad \frac{\partial N}{\partial x} = 0$

$> \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 0$

exact

Integrate M wrt x

2. Integrate M wrt x

$$F_M = \int \frac{1}{x}\,dx = \ln(x) + c_1(y)$$

3. Integrate N wrt y

$$F_N = \int 6y^2\,dy = \frac{6y^3}{3} + c_2(x)$$

4. Combine terms

$$F_M = \underline{\ln(x)} + \underline{c_1(y)}$$

$$F_N = \underline{2y^3} + \underline{c_2(x)}$$

There are no common terms

$$F(x,y) = \ln(x) + 2y^3$$

Solution to the ODE

$$\boxed{\ln(x) + 2y^3 = C}$$

II. Reducible 2nd order ODE:

A general 2nd order ODE can be written:

$$F(x, y, y', y'') = 0$$

GOAL: Make a change of variables to convert this into a 1st order ODE.

Case 1: y is missing from ODE

$$F(x, y', y'') = 0$$

Substitution: $y' = p(x)$

$y'' = p'(x)$

$F(x, p, p') = 0 \longrightarrow$ 1st order

$F(x, \overset{y}{p}, \overset{\prime}{p'}) = 0 \longrightarrow$ 1st order

solve for $p(x)$

↳ convert back to $y(x)$

Ex: $xy'' + 3y' = 8x$ 2nd order linear

y-term is missing

substitute: $y' = p(x)$

$y'' = p'$

$xp' + 3p = 8x$ 1st order

Rewrite:

$p' + \frac{3}{x} p = 8$ linear

solve by integrating factor

$\mu(x) = \exp\left(\int \frac{3}{x} dx\right) = \exp(3\ln(x)) = x^3$

solve for $p(x)$

$x^3 p' + \frac{3}{x} x^3 p = 8x^3$

$\int \frac{d}{dx}\left[x^3 p\right] = \int 8x^3$

$x^3 p = \frac{8x^4}{4} + C_1 = 2x^4 + C_1$

$p(x) = x^{-3}\left[2x^4 + C_1\right]$

$p(x) = 2x + C_1 x^{-3}$

Convert back to $y(x)$ 1st order ODE

$y' = p(x) = 2x + C_1 x^{-3}$

solve by direction integration

$\int y' = \int 2x + C_1 x^{-3} \, dx$

$y(x) = \frac{2x^2}{2} + C_1 \frac{x^{-2}}{-2} + C_2$

$$y(x) = \frac{2x}{2} + c_1 \frac{x}{-2} + c_2$$

Solution:

$$\boxed{y(x) = x^2 - \frac{c_1}{2} x^{-2} + c_2}$$

Recall: $F(x, y, y', y'') = 0$

case 2: x is missing

$$F(y, y', y'') = 0$$

substitution: $y' = p(y)$

$$y'' = \frac{dp}{dx} = \frac{dp}{dy} \overset{p}{\frac{dy}{dx}} = \left(\frac{dp}{dy}\right) p = p \frac{dp}{dy}$$

ODE: $F(y, p, p\frac{dp}{dy}) = 0$

solve for $p(y)$

convert back to $y(x)$

Ex: $y y'' = (y')^2$

x term is missing

substitution: $y' = p$ plug into ODE

$$y'' = p \frac{dp}{dy}$$

ODE:

$$y\left(\cancel{p}\,\frac{dp}{dy}\right) = (p)^{\cancel{2}}$$

$$\frac{dp}{dy} = \frac{p}{y}$$

1st order separable

solve by separation of variables

$$\int \frac{dp}{p} = \int \frac{dy}{y}$$

$$e^{\ln p} = e^{\ln y + C_1} = \left(e^{\ln y}\right)\left(\underbrace{e^{C_1}}_{A}\right)$$

$$p = Ay$$

Convert p(y) into y(x)

$$y' = p = Ay$$

$$\frac{dy}{dx} = Ay$$

1st order ODE separable

Separation of variables

$$\int \frac{dy}{y} = \int A\,dx$$

$$e^{\ln y} = e^{Ax + C_2}$$

$$\boxed{y(x) = Be^{Ax}}$$

NOTE: We have 2 unknown constants

→ 2 integrals

→ 2nd order equation

Summary:

Method

Summary:

Type	Eqn	Method
Exact	$M(x,y)dx + N(x,y)dy = 0$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$	Integrate (F_N, F_M) $F(x,y) = C$
2nd order Reducible (no y term)	$F(x, y', y'') = 0$	Substitution $y' = p(x)$ $y'' = p'$
2nd order Reducible (no x term)	$F(y, y', y'') = 0$	Substitution $y' = p(y)$ $y'' = p\frac{dp}{dy}$