Section 2.1

Population Models

Warm up: Solve $\frac{dP}{dt} = kP$, $\quad P(0) = P_0$

1st order, linear, separable $\longrightarrow$ separation of variables

$$P(t) = P_0 e^{kt}$$

I. General Population Eqn:

In Sec 1.4 we studied

$$\frac{dP}{dt} = kP$$

$P(t)$ - population

k - growth constant

(#births - # deaths)

assumed k was constant

solution $P(t) = P_0 e^{kt}$

Most general form of the population eqn:

$$\frac{dP}{dt} = \beta P - \delta P, \qquad P(0) = P_0$$

where $\beta = \beta(P, t)$ birth rate

$\delta = \delta(P, t)$ death rate

Ex: Consider $\beta(P, t) = 3t^2$ Births increase in time

$\delta(P, t) = 0$ no deaths

Then: $\frac{dP}{dt} = 3t^2 P$, $\qquad P(0) = P_0$

1st order, linear, separable

$\rightarrow$ separation of variables

$$\int \frac{dP}{P} = \int 3t^2 \, dt$$

$$\ln P = \frac{\cancel{3}t^3}{\cancel{3}} + C = t^3 + C$$

$$e \qquad\qquad\qquad e$$

$$P(t) = Ae^{t^3}$$

$$P(t) = Ae^{t}$$

init. cond: $P(0) = P_0 = \left[A e^{t^3} \right]\Big|_{t=0} = A = P_0$

$$\boxed{P(t) = P_0 e^{t^3}}$$ very fast growth

II. Bounded Populations & Logistic Equation:

In nature observed
birth rate (β) decreases as population (P) increases

$$\frac{dP}{dt} = aP - \underbrace{bP^2}_{}$$ decline in birth rate

If $a, b > 0$ then we call this the logistic equation

Sometimes it written as:

$$\frac{dP}{dt} = kP(M - P)$$ $k = b$, $M = \frac{a}{b}$

Ex: $\frac{dP}{dt} = 2P - P^2$ $P(0) = 1$

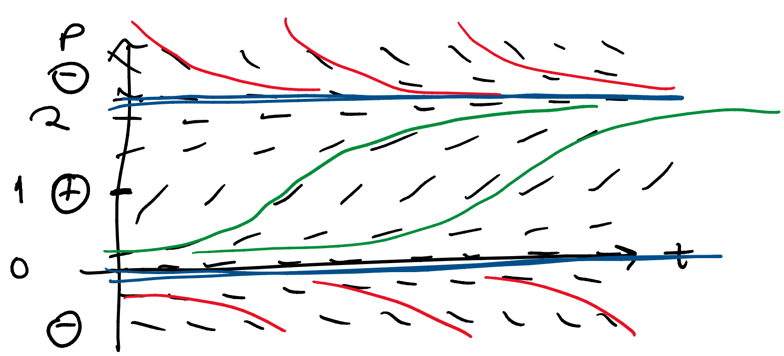
1st order, nonlinear, separable

first draw a slope field — visual picture

$$\frac{dP}{dt} = 2P - P^2 = P(2-P)$$

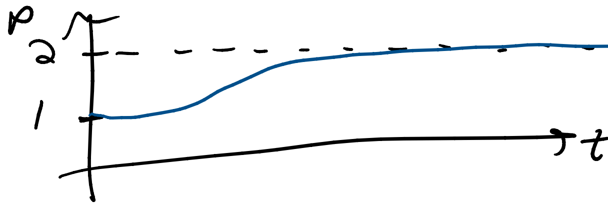
eq solns: $P_*(2 - P_*) = 0 \longrightarrow P_* = 0, \; P_* = 2$

$P_* = 0, \; P_* = 2$ slope $= 0$



... then solution looks like

so if $P(0)=1$, then solution looks like



$\lim_{t\to\infty} P(t) = 2 = P_*$

asymptote at $P=2$

Let's go back + solve using separation of variables

$$\frac{dP}{dt} = P(2-P) \qquad P(0)=1$$

$$\underbrace{\int \frac{dP}{P(2-P)}} = \int dt = t + C$$

To evaluate this integral, use the method of partial fractions to expand this out

Partial Fractions:

$$\frac{1}{P(2-P)} = \frac{A}{P} + \frac{B}{2-P}$$

(reverse of adding fractions)

Need to find A and B

multiply both sides by $P(2-P)$

$$\frac{\cancel{P(2-P)}}{\cancel{P(2-P)}} = \frac{A\cancel{P}(2-P)}{\cancel{P}} + \frac{BP\cancel{(2-P)}}{\cancel{2-P}}$$

$$1 = A(2-P) + BP$$

combine like terms

$$1 = 2A + (B-A)P$$

$P = P(t)$ varies in time

separate out into two eqns

$1 = 2A$

$A = \frac{1}{2}$

$0 = (B-A)P$ (Assume $P(t) \neq 0$)

$0 = B - A$

$B = A = \frac{1}{2}$

So $$\boxed{\frac{1}{P(2-P)} = \frac{1/2}{P} + \frac{1/2}{2-P}}$$ partial fractions

plug this into integral

$$\int \frac{1/2\, dP}{P} + \int \frac{\frac{1}{2} dP}{2-P} = \int \frac{dP}{P(2-P)} = \int dt = t + C$$

$u = 2-P$
$du = -dP$

$$\frac{1}{2}\ln P - \frac{1}{2}\int \frac{du}{u} = t + C$$

$$\frac{1}{2}\ln P - \frac{1}{2}\ln(u) = t + c$$

$$\frac{1}{2}\ln P - \frac{1}{2}\ln(2-P) = t + C$$

$$\frac{1}{2}\ln\left(\frac{P}{2-P}\right) = t + C_1$$

$$e^{\ln\left(\frac{P}{2-P}\right)} = e^{2t + C_2}$$

$$\frac{P}{2-P} = Ae^{2t}$$

$$P = Ae^{2t}(2-P) = 2Ae^{2t} - Ae^{2t}P$$

$$P(1+Ae^{2t}) = P + Ae^{2t}P = 2Ae^{2t}$$

$$P = \frac{2Ae^{2t}}{1+Ae^{2t}}\left(\frac{e^{-2t}}{e^{-2t}}\right)$$

$$P(t) = \frac{2A}{e^{-2t} + A}$$

plug in initial cond: $P(0) = 1 = \left[\frac{2A}{e^{-2t} + A}\right]\Big|_{t=0} = \frac{2A}{1+A}$

$$1 + A = 2A$$
$$1 = A$$

$$\boxed{P(t) = \frac{2}{e^{-2t} + 1}}$$

Q: Does this match our slope field?

@ $t=0$, $\quad P(0) = 1$

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{2}{\underbrace{e^{-2t}}_{\to 0 \text{ as } t\to\infty} + 1} = 2 = P^* \quad \checkmark$$

matches slope field

$\boxed{P^* = 2}$ This is called the limiting population

III. Limiting Population & Carrying Capacity

logistic IVP: $\quad \frac{dP}{dt} = kP(M-P) \quad , \quad P(0) = P_0$

Here: $\quad P_0$ - initial population > 0

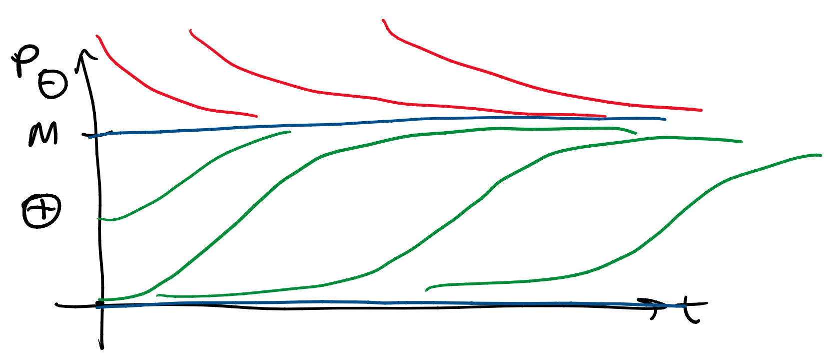
k - constant > 0

M - limiting population > 0
(carrying capacity)

eq solns: $\quad P_* = 0, \quad P_* = M$

slope field

(only consider $P > 0$)



For all P_0, $\quad \lim_{t\to\infty} P(t) = M \quad$ (limiting population)

The "world" wants to hold M people

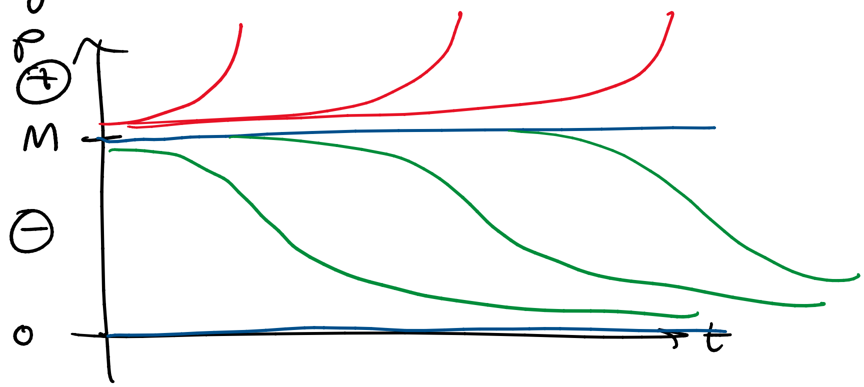
Q: What happens if we change the sign of RHS

$$\frac{dP}{dt} = -kP(M-P) \qquad P(0) = P_0$$

eq solns: stay the same $P^* = 0$, $P_* = M$

slope switches sign

slope field



so now, M is called a threshhold population

If $P_0 < M$ then $\lim_{t \to \infty} P(t) = 0$ population dies out "extinction"

If $P_0 > M$ then $\lim_{t \to \infty} P(t) = +\infty$ population grows to $+\infty$

Summary

Type	Eqn	Method
logistic equation	$\frac{dP}{dt} = aP - bP^2$ $= kP(M-P)$	separation of variables ↳ partial fractions