☆ Equilibrium Solutions & Stability

Warm up: Find the equilibrium solution(s) of:

$$\frac{dx}{dt} = (x-2)(x+3)$$

Ans: $(x-2)(x+3) = 0$ $\quad x_* = 2, -3$ $\quad$ (two eq sol'ns)

I. Qualitative Analysis: $\frac{dx}{dt} = f(x,t) \quad x(0) = x_0$

Expand on the technique of slope fields to develop a qualitative understand

Def: A 1st order ODE is called autonomous if it can be written

$$\frac{dx}{dt} = f(x)$$

no t terms on the RHS

NOTE: 1. This opposite of Sec 1.2

$$\frac{dx}{dt} = g(t)$$

no x terms on RHS

2. An autonomous is also separable

Def: The solutions of $f(x) = 0$ are called critical points

Note: If c is a critical point $(f(c) = 0)$ then the constant function $x(t) = c$ solves the ODE and is called an equilibrium solution.

Ex: logistic equation

$$\frac{dx}{dt} = kx(M-x), \quad x(0) = x_0 \qquad k, M > 0$$

this an autonomous ODE

this ~~is~~ an autonomous ODE

critical points: $f(x) = 0 = kx(M-x)$

$c = 0, M$

so $x_*(t) = 0$ and $x_*(t) = M$ are equilibrium solutions.

II. Stability of Critical Points

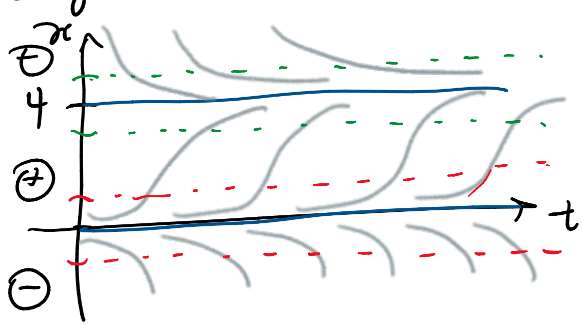
$$\frac{dx}{dt} = f(x) \qquad x(0) = x_0 \qquad (*)$$

Def: A critical point c of $(*)$ is called stable provided that if x_0 is close enough to c, then $x(t)$ remains close to c for all $t > 0$.

Otherwise, the critical point c is called unstable

Ex: $\frac{dx}{dt} = x(4-x)$ crit pts: $c = 0, 4$

Slope field



solution curves look like a funnel
$\Rightarrow$ stable

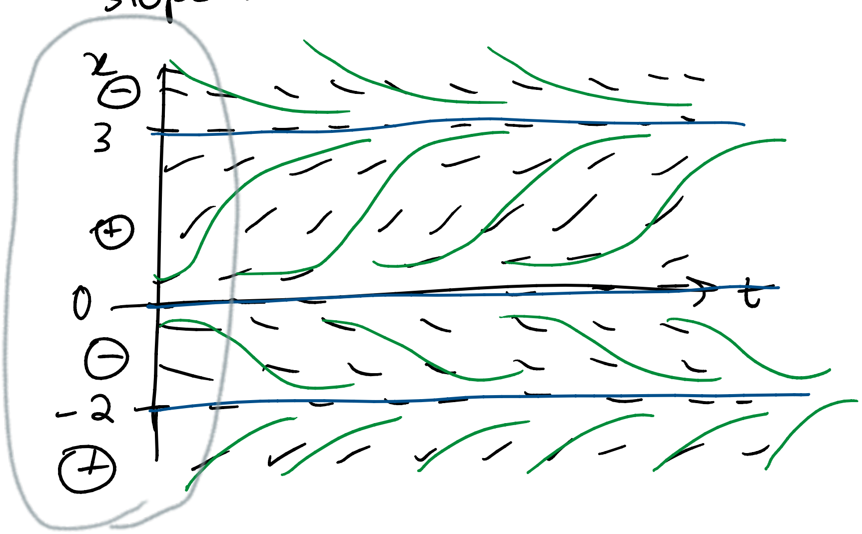
solution curves look like a spout
$\Rightarrow$ unstable

Ex: $\frac{dx}{dt} = x(3-x)(2+x)$

critical points: $x(3-x)(2+x) = 0$

$x = 0, 3, -2$

Slope field



if $x > 3$
$$\frac{dx}{dt} = x(3-x)(2+x) = (+)(-)(+) = (-)$$

if $0 < x < 3$
$$\frac{dx}{dt} = (+)(+)(+) = (+)$$

if $-2 < x < 0$
$$\frac{dx}{dt} = (-)(+)(+) = (-)$$

if $x < -2$
$$\frac{dx}{dt} = (-)(+)(-) = (+)$$

Stability of crit pts:

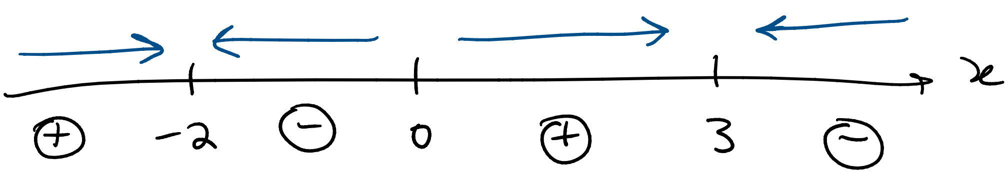
$c = 3$ funnel $\rightarrow$ stable

$c = 0$ spout $\rightarrow$ unstable

$c = -2$ funnel $\rightarrow$ stable

Phase portrait:

Draw only the x-axis



stable unstable stable

III. Harvesting a Logistic Population

$$\frac{dx}{dt} = \underbrace{2x - x^2}_{\text{logistic eqn}} - h \qquad \text{where } h > 0$$

1st order, nonlinear, autonomous

$x(t)$ = population of fish in a lake

h = # of fish removed by fishing

Q: How do the critical points c depend on the harvesting rate h?

Ex: $h = 0$ (no harvesting)

$$\frac{dx}{dt} = x(4-x)$$ so c.p. are $c = 0, 4$

$c = 4$ limiting population

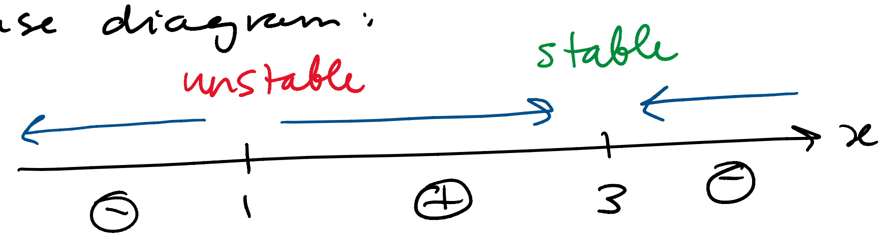
Ex: $h = 3$

$$\frac{dx}{dt} = x(4-x) - 3 = -x^2 + 4x - 3$$

$$\frac{dx}{dt} = -(x-3)(x-1)$$

critical points $c = 3, 1$

phase diagram:



if $x > 3$, $\frac{dx}{dt} = -(x-3)(x-1) = \ominus \oplus \oplus = \ominus$

if $1 < x < 3$ $\frac{dx}{dt} = \ominus \ominus \oplus = \oplus$

if $x < 1$ $\frac{dx}{dt} = \ominus \ominus \ominus = \ominus$

so $c = 3$ is stable c.p.

$c = 1$ is unstable c.p.

(Assume x in units of 100s of fish)

if $x_0 > 100$ fish, then pop stabilize at 300 fish

if $x_0 < 100$ fish, then pop dies out due to over-harvesting

IV. Bifurcation:

A physical system may depend strongly on one (or more) parameters in the eqn

$$\frac{dx}{dt} = x(4-x) - h$$

when $h=3$, c.p are $c=1, 3$

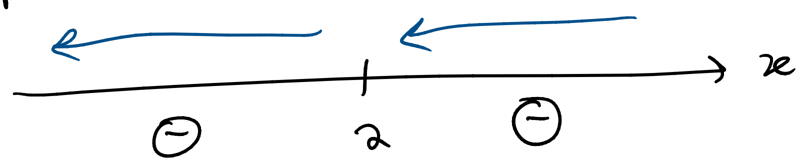
when $h=0$, c.p. are $c=0, 4$

Q: what happens if $h=4$?

$$\frac{dx}{dt} = x(4-x) - 4 = -x^2 + 4x - 4 = -(x-2)^2$$

so the critical points: $c = 2$

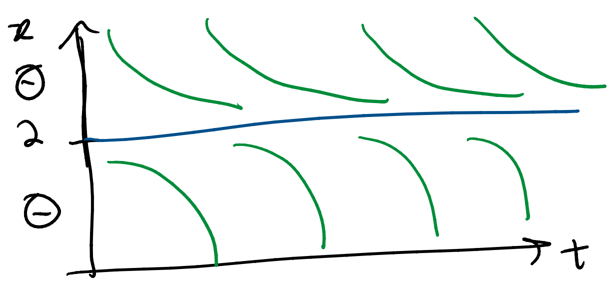
phase portrait:



if $x>2$, $\frac{dx}{dt} = -(x-2)^2 = ⊖ ⊕ ⊕ = ⊖$

if $x<2$, $\frac{dx}{dt} = ⊖ ⊖ ⊖ = ⊖$

slope field



neither a funnel nor a spout

call this "semi-stable"

Q: what if $h > 4$?

$$\frac{dx}{dt} = x(4-x) - h = -x^2 + 4x - h$$

$$x^2 + 4x - h = 0$$

dt

c.p. $\quad -x^2 + 4x - h = 0$

$$x^2 - 4x + h = 0$$

quadratic formula

$$x = \frac{-(-4)}{2\cdot 1} \pm \frac{\sqrt{(-4)^2 - 4(1)(h)}}{2\cdot 1}$$

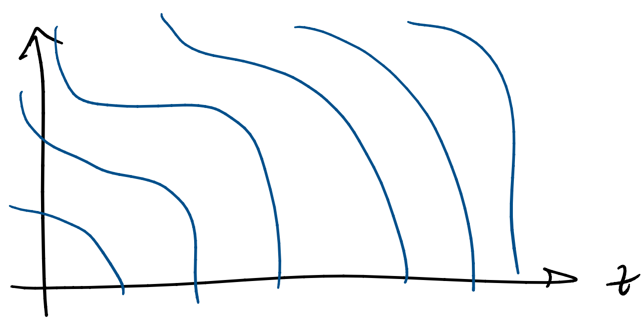
$$= 2 \pm \tfrac{1}{2}\sqrt{16 - 4h}$$

$$\boxed{c = 2 \pm \sqrt{4 - h}}$$

general formula for c.p.

if $h > 4$, then $c = 2 \pm i\omega$ is complex-valued

soln curves



no (real-valued) crit points

no equilibrium solns

for every x_0 the fish pop dies out

→ too much harvesting

So for this fishing problem:

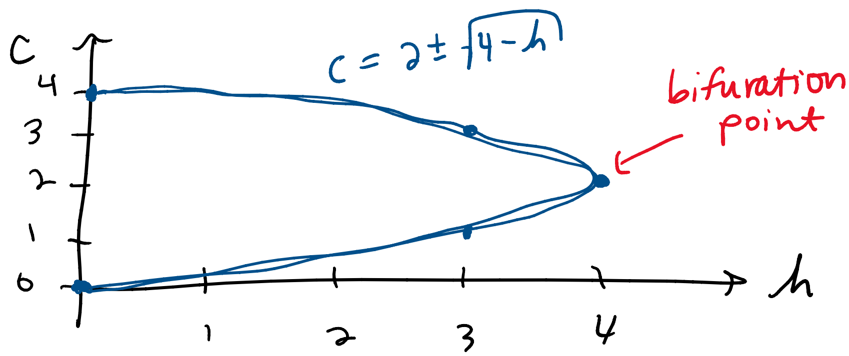
$0 \le h < 4 \longrightarrow$ two critical points

$h = 4 \longrightarrow$ one critical point

$h > 4 \longrightarrow$ no critical points

The point $h = 4$ where the qualitative behavior changes is called the <u>bifurcation point</u>

Show this visually by plotting a <u>bifurcation diagram</u>



general soln for crit. point

$c = 2 \pm \sqrt{4-h}$

@ $h = 0$ $\quad c = 0, 4$

@ $h = 3$ $\quad c = 1, 3$

@ $h = 4$ $\quad c = 2$

@ $h = 0$ $\quad$ no c

Summary

Type	Eqn	Method
Antonomous	$\frac{dx}{dt} = f(x)$	slope field phase portrait critical points stability
Harvesting	$\frac{dx}{dt} = f(x) - h$	same tools ↑ bifurcation point bifurcation diagram