Section 2.3

☆ Velocity - Acceleration Models

Warm up: what is the position $x(t)$ for a particle moving with constant acceleration a? $x'' = a$

Ans: $x(t) = \frac{1}{2}at^2 + v_0 t + x_0$

I. Velocity - Acceleration

position $x(t)$ when acceleration is constant

$$x(t) = \frac{1}{2}at^2 + v_0 t + x_0$$

v_0 - initial vel $\quad$ x_0 - initial position

Ex: We shoot an arrow straight up from the ground ($x_0 = 0$) and with initial velocity $v_0 = 49\,m/s$

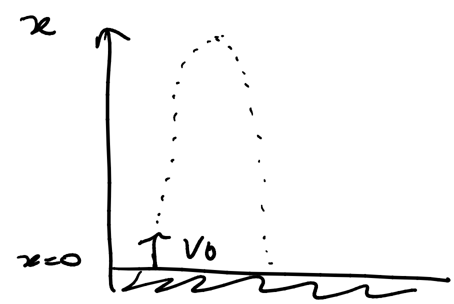
(a) what is the max height the arrow reaches?

(b) At what time does the arrow hit the ground?

arrow pulled by gravity

$a = g = -9.8\ m/s^2$

$x_0 = 0 \qquad v_0 = 49$



plug into eqn

$$x(t) = -\frac{9.8t^2}{2} + 49t + 0$$

$$\boxed{x(t) = -4.9t^2 + 49t}$$

(a) arrow reach the max height when $v(t_m) = 0$

$$v(t) = at + v_0 = -9.8t + 49$$

$$0 = v(t_m) = -9.8t_m + 49$$

$$t_m = \frac{49}{9.8} = 5\,s$$

$$x_{max} = x(t_m) = \left[-4.9t^2 + 49t\right]\Big|_{t=5}$$
$$= -4.9(5)^2 + 49(5)$$
$$\boxed{x_{max} = 122.5\text{ m}}$$

(b) The arrow hits ground when

$$x(t_g) = 0 = \left[-4.9t_g^2 + 49t_g\right]$$
$$0 = -4.9t_g\left[t_g - 10\right]$$
$$t_g = 0, 10$$
$$\boxed{t = 10s}$$

Q: What happens when there is air resistence?

Let F_R be the force exerted by air resistence

↳ air resistence always opposes velocity

Newton's 2nd law:

$$m\frac{dv}{dt} = F_G + F_R$$

Simple conditions: $F_R = kv^2$

other conditions: $F_R = kv^p$ where $1 \le p \le 2$

and k depends on the shape of the body

Case $p=1$: body mass m

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$$F_G = -mg \qquad F_R = -kv \qquad k>0$$

Newton's 2nd law:

$$m\frac{dv}{dt} = -kv - mg$$

divide both sides by m

$$\frac{dv}{dt} = -\rho v - g \qquad \text{where } \rho = \frac{k}{m} > 0$$

ρ is called the drag coefficient

IVP: $\dfrac{dv}{dt} = -\rho v - g \qquad v(0) = v_0$

1st order, linear, separable, autonomous

separation of variables

$u = \rho v + g$
$du = \rho\, dv$

$$\int \frac{dv}{\rho v + g} = \int -dt$$

$$\frac{1}{\rho}\int \frac{du}{u} = \int -dt$$

$$\frac{1}{\rho}\ln u = -t + C_1$$

$$e^{\ln u} = e^{-\rho t + C_2}$$

$$\rho v + g = u = Ae^{-\rho t}$$

$$v(t) = \frac{1}{\rho}\left(Ae^{-\rho t} - g\right)$$

init cond. $v(0) = v_0 = \dfrac{1}{\rho}\left(A\cancel{e^{-\rho 0}}^{\,1} - g\right)$

$$\rho v_0 = A - g$$

$$A = \rho v_0 + g$$

$$v(t) = \frac{1}{\rho}\left((\rho v_0 + g)\, e^{-\rho t} - g \right)$$

$$\boxed{v(t) = \left(v_0 + \frac{g}{\rho}\right) e^{-\rho t} - \frac{g}{\rho}}$$

Note that

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \left(v_0 + \frac{g}{\rho}\right)\underbrace{e^{-\rho t}}_{\substack{\text{exp decay} \\ \to 0}} - \frac{g}{\rho} = \frac{-g}{\rho}$$

The speed of a falling body approaches a finite limiting speed called the terminal velocity V_t

The terminal speed

$$|V_t| = \left|\frac{-g}{\rho}\right| = \frac{mg}{k}$$

Rewrite

$$\boxed{v(t) = (v_0 + V_t)\, e^{-\rho t} + V_t}$$

To get height, integrate $v(t)$

$$x(t) = \int v(t)\,dt = \int \left((v_0 + V_t) e^{-\rho t} + V_t \right) dt$$

$$= -\frac{1}{\rho}(v_0 + V_t) e^{-\rho t} + V_t t + C$$

plug in init cond.

$$x(0) = x_0 = -\frac{1}{\rho}(v_0 + V_t)\cancelto{1}{e^{0}} + \cancelto{0}{V_t\, 0} + C$$

$$C = x_0 + \frac{1}{\rho}(v_0 + V_t)$$

so $$\boxed{x(t) = x_0 + V_t t + \frac{1}{\rho}(v_0 + V_t)\left[1 - e^{-\rho t}\right]}$$

so $$\boxed{x(t) = \underbrace{x_0 + v_\tau t}_{\text{these terms look similar}} + \frac{1}{\rho}(v_0 + v_\tau)\underbrace{\left[1 - e^{-\rho t}\right]}_{\text{exp instead of } t^2}}$$

Arrow example:

$V_0 = 49\,m/s \qquad x_0 = 0 \qquad g = -9.8$

Assume $\rho = 0.04 = \frac{1}{25}$

then $V_\tau = \frac{g}{\rho} = \frac{-9.8}{1/25} = -245$

then $v(t) = (v_0 - v_\tau)e^{-\rho t} + v_\tau$

$$\boxed{v(t) = 294\,e^{-t/25} - 245}$$

and the position

$$x(t) = x_0 + v_\tau t + \frac{1}{\rho}(v_0 - v_\tau)\left[1 - e^{-\rho t}\right]$$

$$= 0 + (-245)t + 25(294)\left[1 - e^{-t/25}\right]$$

$$\boxed{x(t) = 7350 - 245t - 7350\,e^{-t/25}}$$

(a) Arrow reach max height when

$$v(t_m) = 0$$

$$294\,e^{-t_m/25} - 245 = 0$$

$$e^{-t_m/25} = \frac{245}{294}$$

$$t_m = 25 \ln\left(\frac{294}{245}\right) \approx 4.86\,s$$

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$$x_{max} = x(t_m) \approx \boxed{108.2\ m}$$

(b) arrow hits the ground at
$x(t_g) = 0$

$$7350 - 245 t_g - 7350 e^{-t_g/25} = 0$$

solve this numerically (use Matlab, Wolfram Alpha, etc...)

$$\boxed{t_g \approx 9.41\ s}$$

Comparison

	No Resistence	Air Resistence
t_m	5s	4.86s
x_m	122.5 m	108.2 m
t_g	10s	9.41 s

the arrow doesn't go as high and it hits the ground sooner.