Section 3.1

☆ Introduction to 2nd order Linear Equations

Warm up: What is the solution to:

$$y' - ry = 0 \; ? \qquad \text{where } r \text{ is a const}$$

1st order, linear, rewrite

$$y' = ry \quad \longrightarrow \text{separable}$$

$$\int \frac{dy}{y} = \int r\,dx$$

$$e^{\ln y} = e^{rx + C}$$

$$y(x) = Ae^{rx}$$

I. 2nd Order Linear Eqn:

Def: A 2nd order Linear ODE can be written in the form:

$$(*) \qquad A(x)y'' + B(x)y' + C(x)y = F(x)$$

If $F(x) \equiv 0$, then the ODE is called homogeneous

- 2nd order
- linear

Otherwise, if $F(x) \not\equiv 0$, then it is called non-homogeneous

Note: Unfortunately 2 definitions of homogeneous

(vs. $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$)
- 1st order
- nonlinear

BUT – use context clues to determine which

Special case:

Assume: $A(x) = a$, $B(x) = b$, $C(x) = c$ } all constants, and $F(x) = 0$

Then:

$$ay'' + by' + cy = 0 \qquad (\square)$$

... say that $(\square)$ has constant coefficients

$ay'' + by' + cy = 0$

we say that (□) has <u>constant</u> <u>coefficients</u>

<u>Strategy</u>: Look for solutions of the form
$y(x) = e^{rx}$ — r constant unknown.

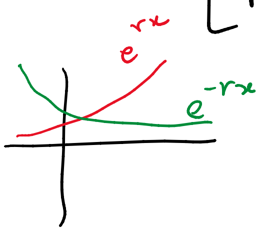
$a = 0 \quad c = -8$
$b = +2$

<u>Ex</u>: $y'' + 2y' - 8y = 0$

<u>Anstatz</u>: $y = e^{rx}$
plug into the ODE
$y' = re^{rx} \qquad y'' = r^2 e^{rx}$

$$[r^2 e^{rx}] + 2[re^{rx}] - 8[e^{rx}] = 0$$

pull out e^{rx}



$$\underbrace{e^{rx}}_{\text{never zero}} \underbrace{[r^2 + 2r - 8]}_{\text{must be zero}} = 0$$

$$\boxed{r^2 + 2r - 8 = 0}$$

called the <u>characteristic</u> <u>equation</u> of the ODE
Find the roots of char. eq:

$$(r+4)(r-2) = 0$$

Roots: $r = -4$ and $r = +2$

Solutions had the form $y(x) = e^{rx}$
when $r = -4$, we get one solution
$y_1(x) = e^{-4x}$

Check that y_1 solves the ODE:
$y_1 = e^{-4x} \qquad y_1' = -4e^{-4x} \qquad y_1'' = 16e^{-4x}$

Check that ...

$y_1 = e^{-4x}$ $\quad y_1' = -4e^{-4x}$ $\quad y_1'' = 16e^{-4x}$

plug into ODE

$$y'' + 2y' - 8y \stackrel{?}{=} 0$$

$$[16e^{-4x}] + 2[-4e^{-4x}] - 8[e^{-4x}] \stackrel{?}{=} 0$$

$$e^{-4x}[16 - 8 - 8] \stackrel{?}{=} 0$$

$$0 = 0 \checkmark$$

so yes $y_1(x)$ solves the ODE

Similarly, when $r = +2$

$y_2(x) = e^{2x}$ solves the ODE.

The <u>general solution</u> to the ODE is a linear combination of the 2 solutions

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$\boxed{y(x) = c_1 e^{-4x} + c_2 e^{2x}}$$

<u>Principle of Superposition</u>:

If $y_1(x)$ and $y_2(x)$ solve $(*)$ then $y(x) = c_1 y_1(x) + c_2 y_2(x)$, where c_1 and c_2 are constants, also solves $(*)$

<u>Note</u>: Only true if the ODE is <u>linear</u>

<u>Ex</u>: Consider the IVP:

$$y'' + 2y' - 8y = 0, \quad y(0) = 3, \quad y'(0) = -7$$

Note: A 2nd order ODE will have 2 initial conditions

Note: A 2nd order ODE will have 2 initial conditions

our general solution

$$y(x) = c_1 e^{-4x} + c_2 e^{2x}$$

1st initial condition

$$y(0) = 3 = \left[c_1 e^{-4x} + c_2 e^{2x}\right]\Big|_{x=0}$$

$$3 = c_1 + c_2$$

2nd initial condition

calc y': $y'(x) = -4c_1 e^{-4x} + 2c_2 e^{2x}$

$$y'(0) = -1 = \left[-4c_1 e^{-4x} + 2c_2 e^{2x}\right]\Big|_{x=0}$$

$$-1 = -4c_1 + 2c_2$$

So now, need to solve:

$$\begin{cases} 3 = c_1 + c_2 \\ -1 = -4c_1 + 2c_2 \end{cases}$$

system of linear algebraic equation

2 equations
2 unknowns
has a unique soln

$$c_1 = 3 - c_2$$

$$-1 = -4(3 - c_2) + 2c_2$$

$$-1 = -12 + 4c_2 + 2c_2 = -12 + 6c_2$$

$$11 = 6c_2 \qquad c_2 = \frac{11}{6}$$

$$c_1 = 3 - \frac{11}{6} = \frac{18-11}{6} = \frac{7}{6}$$

particular solution

$$\boxed{y(x) = \frac{7}{6} e^{-4x} + \frac{11}{6} e^{2x}}$$

Here: started w/ ODE $\longrightarrow$ solution y_1 and y_2

go in this direction

Ex: $y(x) = c_1 e^{2x} + c_2 e^{-x}$

Q: what ODE does it solve?

roots of characteristic eqn: $r=2,\ r=-1$

derive char eqn: $(r-2)(r+1) = 0$

expand this out $r^2 + r - 2r - 2 = 0$

char. eqn: $r^2 - r - 2 = 0$

$$\underbrace{(1)}_{a} r^2 + \underbrace{(-1)}_{b} r + \underbrace{(-2)}_{c} = 0$$

plug into (☆)

$$ay'' + by' + cy = 0$$

This an ODE that has that solution $\boxed{y'' - y' - 2y = 0}$

NOT necessarily unique

multiply by 3

$$3y'' - 3y' - 6y = 0$$

has the same solution.

Recall

$$ay'' + by' + cy = 0$$

Procedure:

1. Find the characteristic equation
$$ar^2 + br + c = 0$$

2. Find the roots r_1 and r_2

(*) ... two linearly independent (*) solutions

2. Find the ...

(*) more detail next lecture

3. Write down two linearly independent (*) solutions

$y_1(x) = e^{r_1 x}$ $\qquad$ $y_2 = e^{r_2 x}$

4. Use principle of superposition

general soln: $\quad y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Q: What if the characteristic equation has a repeated root? $(r_1 = r_2)$?

Ex: $\quad y'' + 2y' + y = 0$

1. characteristic eqn:

$$r^2 + 2r + 1 = 0$$

2. Find the roots

$$(r+1)^2 = 0$$

$r_1 = -1$ $\qquad$ repeated root

$r_2 = -1$

3. We have at least 1 solution

$$y_1(x) = e^{-x}$$

but, the 2nd solution is not linearly independent

$$y_2(x) \cancel{=} e^{-x}$$

Q: What happens if we take $y_2(x) = e^{-x}$

$y'' + 2y + y = 0, \qquad y(0) = 1, \; y'(0) = 2$

Let the general soln:

$$y(x) = c_1 e^{-x} + c_2 e^{-x}$$

$$y'(x) = -c_1 e^{-x} - c_2 e^{-x}$$

initial conditions

$x \qquad -x]1$

initial conditions

$$y(0) = 1 = \left[c_1 e^{-x} + c_2 e^{-x}\right]\Big|_{x=0}$$
$$1 = c_1 + c_2$$

$$y'(0) = 2 = \left[-c_1 e^{-x} - c_2 e^{-x}\right]\Big|_{x=0}$$
$$2 = -c_1 - c_2$$

want to solve:

$$1 = c_1 + c_2$$
$$2 = -c_1 - c_2$$

$$c_1 = 1 - c_2$$

$$2 = -[1 - c_2] - c_2 = -1 + \cancel{c_2} - \cancel{c_2}$$
$$2 = -1 \implies\Leftarrow$$

cannot find c_1 and c_2 to solve the IVP.

<u>Instead</u>: let's take

$$y_2(x) = x e^{-x} \quad \longleftarrow \text{multiply by } x$$

<u>Exercise</u>: Check that y_2 solves the ODE

General soln: $\quad y(x) = c_1 e^{-x} + c_2 x e^{-x}$

plug in init cond: $\quad y(0) = 1, \quad y'(0) = 2$

$$y'(x) = -c_1 e^{-x} + c_2 e^{-x} - c_2 x e^{-x}$$
$$y'(x) = (c_2 - c_1) e^{-x} - c_2 x e^{-x}$$

$$y(0) = 1 = \left[c_1 e^{-x} + c_2 x e^{-x}\right]\Big|_{x=0}$$
$$1 = c_1 + 0 \longrightarrow c_1 = 1$$

$$y'(0) = 2 = \left[(c_2 - c_1)e^{-x} - c_2 x e^{-x}\right]\Big|_{x=0}$$

$$2 = c_2 - c_1 + 0 = c_2 - 1$$

$$c_2 = 3$$

so the particular solution

$$\boxed{y(x) = e^{-x} + 3xe^{-x}}$$

Check that $y_2(x) = xe^{-x}$ solves $y'' + 2y' + y = 0$

$$y_2' = (1)e^{-x} + x(-e^{-x}) = (1-x)e^{-x}$$

$$y_2'' = (-1)e^{-x} + (1-x)(-e^{-x})$$

$$= [-1-1+x]e^{-x} = [-2+x]e^{-x}$$

plug into ODE

$$[(-2+x)e^{-x}] + 2[(1-x)e^{-x}] + [xe^{-x}] \stackrel{?}{=} 0$$

$$e^{-x}[-2+x+2(1-x)+x] \stackrel{?}{=} 0$$

$$e^{-x}[-\cancel{2}+\cancel{x}+\cancel{2}-\cancel{2x}+\cancel{x}] \stackrel{?}{=} 0 \checkmark$$

$$0 = 0$$

so yes, $y_2(x) = xe^{-x}$ solves the ODE.

II. Linear Independence:

Def: Two functions $y_1(x)$ and $y_2(x)$ are linearly independent if, for k_1 and k_2 constants

$$k_1 y_1(x) + k_2 y_2(x) = 0$$

implies that $k_1 = k_2 = 0$.

Conversely, $y_1(x)$ and $y_2(x)$ are linearly ... exists $k_1 \neq 0$

Conversely, $y_1(x)$ and $y_2(x)$ are <u>linearly dependent</u> if there exists $k_1 \neq 0$ and $k_2 \neq 0$ such that

$$k_1 y_1(x) + k_2 y_2(x) = 0 .$$

<u>Ex</u>: linearly dependent

(a) $y_1(x) = x \qquad y_2(x) = 3x$

$$k_1 y_1 + k_2 y_2 = 0$$

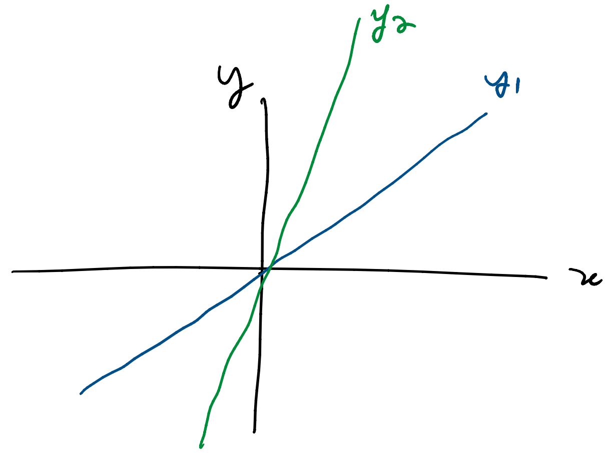
$$k_1 x + k_2(3x) = 0$$

$$(k_1 + 3k_2)x = 0$$

choose $k_2 = 1$, $k_1 = -3$

$$-3x + (1)3x = 0$$

$\longrightarrow$ linearly dependendent



y_2 looks like y_1 stretched

(b) $y_3(x) = e^x \qquad y_4 = \frac{1}{2}e^x$

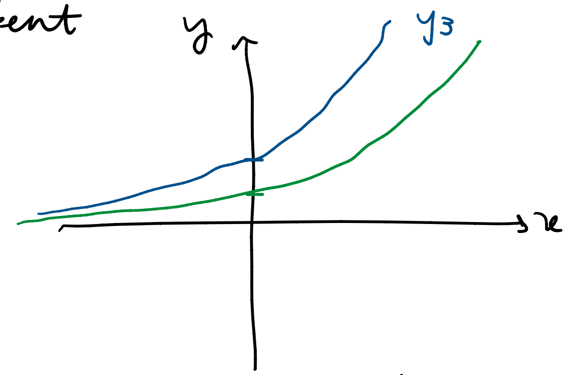
$$k_3 e^x + k_4\left(\tfrac{1}{2}e^x\right) = 0$$

$$e^x\left[k_3 + \frac{k_4}{2}\right] = 0$$

$$k_4 = 2 \qquad k_3 = -1$$

$$-e^x + 2\left(\tfrac{1}{2}e^x\right) = 0$$

so $\longrightarrow$ linearly dependent



y_4 looks like y_3 squished

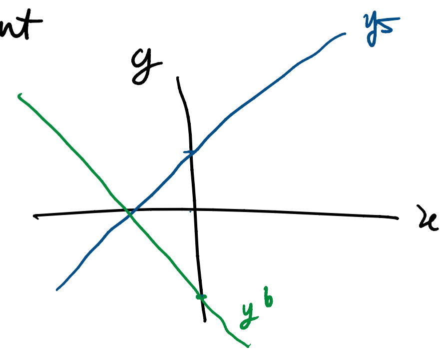
(c) $y_5 = 1 + x \qquad y_6 = -2 - 2x$

$$k_5(1+x) + k_6(-2-2x) \stackrel{?}{=} 0$$

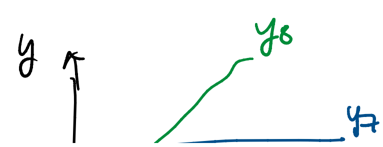
$$y_6 = -2y_5$$

$$k_6 = 1 \qquad k_5 = -2$$

$\longrightarrow$ linearly dependent

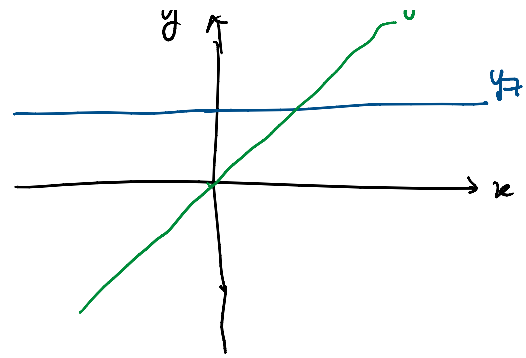


<u>Ex</u>: Linearly Independent:



Ex: Linearly independent:

(d) $y_7 = 1$ $\qquad y_8 = x$



$k_7(1) + k_8 x \stackrel{?}{=} 0$

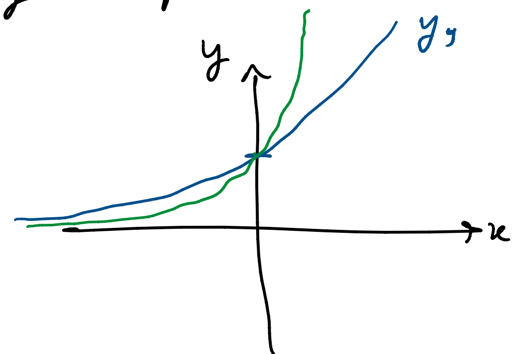
$k_7 = -k_8 x \Rightarrow \Leftarrow$

no longer a constant

$\rightarrow$ need $k_7 = 0$ and $k_8 = 0$

$\hookrightarrow$ linearly independent

(e) $y_9 = e^{x}$ $\qquad y_{10} = e^{2x}$



$k_9 e^{x} + k_{10} e^{2x} \stackrel{?}{=} 0$

$e^{x}\left[k_9 + k_{10} e^{x}\right] \stackrel{?}{=} 0$

$k_9 = -k_{10} e^{x} \Rightarrow \Leftarrow$

no longer a constant

$k_9 = 0$ and $k_{10} = 0$

$\hookrightarrow$ linearly independent

linearly dependent

if $y_1 = k y_2$

constant multiple

linearly independent

if $y_1 \neq k y_2$

more than a constant multiple

Back to Example:

$y_1 = e^{-x}$ $\qquad y_2 = e^{-x}$

$y_1 = (1) y_2 \rightarrow$ linearly dependent

Fix:

$y_1 = e^{-x}$ $\qquad y_2 = x e^{-x}$

$k_1 e^{-x} + k_2 x e^{-x} \stackrel{?}{=} 0$

$k_1 e$ ...

$$e^{-x}\left[k_1 + k_2 x\right] \stackrel{?}{=} 0$$

$$k_1 = -\underbrace{k_2 x}_{\text{no longer a constant}} \Rightarrow \Leftarrow$$

so $k_1 = 0$ and $k_2 = 0$

$\hookrightarrow$ y_1 and y_2 are linearly independent.