☆ General Solutions of Linear Equations

Warm up: Recall the definition of linear independence

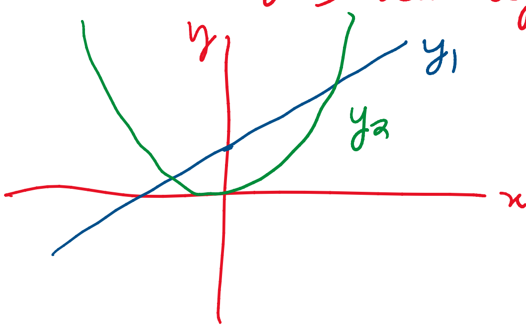
Def: Two functions $y_1(x)$ and $y_2(x)$ are *linearly independent* if

$$k_1 y_1(x) + k_2 y_2(x) = 0$$

where k_1 and k_2 are constants, implies that $k_1 = k_2 = 0$

Plot the functions $y_1(x) = 1+x$ and $y_2(x) = x^2$

are they linearly independent?



y_2 is not a constant multiple of y_1

$$k_1(1+x) + k_2 x^2 = 0$$

$$k_1 = -k_2\left(\frac{x^2}{1+x}\right)$$ this is not a constant

So yes, y_1 and y_2 are linearly indep.

I. Linear Independence:

Recall: $ay'' + by' + cy = 0$ (where r_1 and r_2 are roots of the char. eq^n)

The general solution

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

only if $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are *linearly independent*

Quick check for linear independence:

Def: the *Wronskian* of two functions $f(x)$ and $g(x)$ is

$$W(f,g) = \det\begin{bmatrix} f(x) & g(x) \\ \end{bmatrix} = f(x)g'(x) - f'(x)g(x)$$

Def: the Wronskian of two ...

$$W(f,g) = \det \begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix} = f(x)g'(x) - f'(x)g(x)$$

If $W(f,g) = 0 \longrightarrow f, g$ linearly dependent

If $W(f,g) \neq 0 \longrightarrow f, g$ linearly independent.

Ex: $f(x) = 1+x \qquad g(x) = x^2$

$$W(f,g) = \det \begin{bmatrix} 1+x & x^2 \\ 1 & 2x \end{bmatrix} = (1+x)(2x) - (1)x^2$$

$$= 2x + 2x^2 - x^2 = 2x + x^2 \neq 0$$

so f, g are linearly independent

II. N^{th} order Linear ODE:

$$(*) \quad P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \dots + P_{n-1}(x)y' + P_n(x)y = F(x)$$

In general, what works for 2nd order linear works for n'th order

Principle of superposition: if $y_1, \dots, y_n$ solve $(*)$ then

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x)$$

also solves the ODE.

Ex: $y''' + 3y'' + 4y' + 12y = 0$

has solutions $\quad y_1(x) = e^{-3x}$

$\qquad\qquad\qquad y_2(x) = \cos(2x)$

has solutions $y_1 \ldots$

$$y_2(x) = \cos(2x)$$
$$y_3(x) = \sin(2x)$$

$$y(x) = c_1 e^{-3x} + c_2 \cos(2x) + c_3 \sin(2x)$$
solves the ODE

↳ This is the general solution if y_1, y_2, y_3 are linearly independent.

To check linear independence, compute the Wronskian of 3 fcns.

$$W = \det \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix}$$

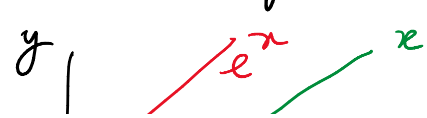
Ex: Show that 1, x, and e^x are linearly independent.

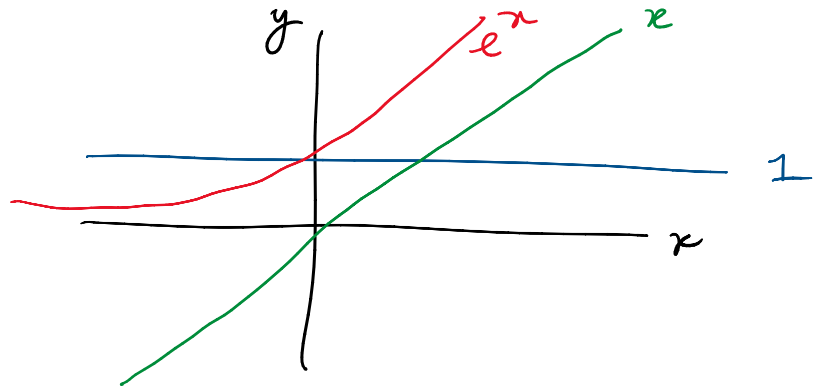
$$W = \det \begin{bmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{bmatrix}$$ expansion by minors

$$= 1 \cdot \det \begin{bmatrix} 1 & e^x \\ 0 & e^x \end{bmatrix} - x \det \begin{bmatrix} 0 & e^x \\ 0 & e^x \end{bmatrix} + e^x \det \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= 1 \cdot (1 \cdot e^x - 0) - \cancel{x(0-0)} + \cancel{e^x[0-0]}$$

$$= e^x \neq 0$$ so yes, linearly independent





Ex: IVP: $y''' - y'' = 0$

$y(0) = 1$, $y'(0) = 2$, $y''(0) = 3$

NOTE: an n^{th} order linear ODE will have n initial conditions

1. characteristic eqn: e^{rx}

$$r^3 e^{rx} - r^2 e^{rx} = 0$$

$$r^3 - r^2 = 0$$

$$r^2(r-1) = 0$$

2. Roots: $r = 1$, $r = 0 \longleftarrow$ repeated root

3. Write down 3 linearly independent solutions

$y_1(x) = e^x \qquad y_2 = e^{0 \cdot x} = 1$

Rule of Thumb: when you have a repeated root, multiply soln by x

Then $y_3(x) = x \cdot (1) = x$

Check that e^x, 1, x are linearly indep. ✓

4. General soln:

$$y(x) = c_1 e^{x} + c_2 + c_3 x$$

5. plug in initial conditions

$$y(0) = 1 = \left[c_1 e^{x} + c_2 + c_3 x\right]\Big|_{x=0}$$

$$1 = c_1 + c_2$$

$$y'(0) = 2 = \left[c_1 e^{x} + c_3\right]\Big|_{x=0}$$

$$2 = c_1 + c_2$$

$$y''(0) = 3 = \left[c_1 e^{x}\right]\Big|_{x=0}$$

$$3 = c_1$$

system of eqns:

$$\begin{cases} 1 = c_1 + c_2 \\ 2 = c_1 + c_3 \\ 3 = c_1 \end{cases}$$

3 eqns
3 unknowns

$$2 = [3] + c_3 \longrightarrow c_3 = -1$$

$$1 = [3] + c_2 \longrightarrow c_2 = -2$$

particular soln:

$$\boxed{y(x) = 3e^{x} - 2 - x}$$

III Method of Reduction of Order

let's go back to: $y'' + 2y' + y = 0$ $\quad -x$

let's go back to: $y'' + 2y' + y = 0$

we found one solution $y_1(x) = e^{-x}$

use the method of reduction of order

guess soln $y_2(x) = v(x)\, y_1(x)$

where $v(x)$ is unknown

plug into the ODE and solve for $v(x)$

$$y_2(x) = v(x) e^{-x}$$

$$y_2'(x) = v'(x) e^{-x} + v(x)[-e^{-x}]$$

$$= [v' - v] e^{-x}$$

$$y_2''(x) = [v'' - v'] e^{-x} + [v' - v](-e^{-x})$$

$$= [v'' - v' - v' + v] e^{-x}$$

$$= [v'' - 2v' + v] e^{-x}$$

plug into ODE:

$$y'' + 2y' + y = 0$$

$$[v'' - 2v' + v] e^{-x} + 2[v' - v] e^{-x} + v e^{-x} = 0$$

$$e^{-x}\left[v'' \cancel{- 2v'} \cancel{+ v} \cancel{+ 2v'} \cancel{- 2v} \cancel{+ v}\right] = 0$$

$$\underbrace{e^{-x}}_{\text{never zero}} v'' = 0$$

$$\rightarrow \quad v'' = 0$$

$$\int v''(x) = \int 0 \, dx$$

$$\int v' = \int c_1$$

$$\int v' = \int c_1$$

$$v(x) = c_1 x + c_2$$

so, $y_2(x) = (c_1 x + c_2) e^{-x}$ is also a soln to the ODE

For simplicity, $c_1 = 1$ and $c_2 = 0$

$y_1(x) = e^{-x}$ $\qquad$ $y_2(x) = x e^{-x}$

Use Wronskian to check linear indep.

$$W = \det \begin{bmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x} - x e^{-x} \end{bmatrix}$$

$$= e^{-x}\left(e^{-x} - x e^{-x}\right) - x e^{-x}\left(-e^{-x}\right)$$

$$= e^{-2x} - \cancel{x e^{-2x}} + \cancel{x e^{-2x}} = e^{-2x} \neq 0$$

so yes, they linearly indep.