Section 3.3 - Part 1

☆ Homogeneous Equations with Constant Coefficients:

Warm up: Find the characteristic equation of the ODE:

$$y''' - 3y'' + 7y' - 8y = 0$$

Ans: substitute $y = e^{rx}$

characteristic eqn: $\boxed{r^3 - 3r^2 + 7r - 8 = 0}$

I. Constant Coefficients:

A n-th order linear homogeneous ODE with constant coefficients can be written:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = 0$$

where $a_0, a_1, \dots, a_n$ are constants and $a_n \neq 0$

Procedure:

1. Guess solutions $y = e^{rx}$
   plug into the ODE to get char. eqn.

   NOTE: $y^{(k)} = r^k e^{rx}$ kth derivative pulls out r^k

   characteristic eqn:

   $$a_n r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0 = 0$$

2. Find the roots of the char. eq.

   NOTE: Fundamental Theorem of Algebra
   every nth order polynomial has n roots

   However, the roots may not be distinct or even real-valued.

quadratic polynomial

For 2nd order ODE $\longrightarrow$ quadratic polynomial

$$ar^2 + br + c = 0$$

use the quadratic formula

For 3+ order ODE

- try factoring ✓
- (may need a computer - outside the scope of MA266)

Characteristic eqn $\longrightarrow$ Roots

3 possible cases

part 1

1. distinct real roots
   $r_1, r_2, \ldots, r_n$

2. repeated real root
   char. eqn: $(r - r_0)^k$

part 2

3. complex-valued roots

<u>Case 1</u>: Distinct Real-Valued Roots

$r_1, r_2, \ldots, r_n$

The solutions

$$\underbrace{y_1 = e^{r_1 x}, \quad y_2 = e^{r_2 x}, \quad \ldots, \quad y_n = e^{r_n x}}_{\text{all linearly independent}}$$

General solution

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \ldots + C_n e^{r_n x}$$

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Ex: $y^{(3)} + 3y'' - 10y' = 0$

characteristic eqn: $r^3 + 3r^2 - 10r = 0$

factor: $r(r^2 + 3r - 10) = 0$

$r(r+5)(r-2) = 0$

real distinct roots: $r = 0, -5, +2$

General solution:

$$\boxed{y(x) = C_1 + C_2 e^{-5x} + C_3 e^{2x}}$$

Case 2: Repeated Real-Valued Roots:

Ex: $y^{(3)} - 3y'' + 3y' - y = 0$

characteristic eqn:

$r^3 - 3r^2 + 3r - 1 = 0$

$(r-1)^3 = 0$

Repeated Root $r = 1$ (triple)

Def: Suppose a characteristic equation has a repeated root $(r - r_0)^k = 0$

Here, the number k is called the multiplicity of the root

(# of times the root is repeated)

In this example: $(r-1)^3 = 0$

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so the root $r=1$ has multiplicity $k=3$

The corresponding solutions

$$y_1 = e^x \qquad y_2 = xe^x \qquad y_3 = x^2e^x$$

General solution:

$$y(x) = C_1e^x + C_2xe^x + C_3x^2e^x$$

$$= \left[C_1 + C_2x + C_3x^2\right]e^x$$

Comment: We can verify this by the Method of Reduction of Order.

NOTE: An ODE can have a combination of repeated and distinct roots.

Ex: $$9y^{(5)} - 6y^{(4)} + y^{(3)} = 0$$

characteristic eqn:

$$9r^5 - 6r^4 + r^3 = 0$$

factor: $$r^3(9r^2 - 6r + 1) = 0$$

$$r^3(3r-1)^2 = 0$$

Roots:

$r=0$ with multiplicity $k=3$

$r=\frac{1}{3}$ with multiplicity $k=2$

General solution

$$y(x) = [c_1 + c_2 x + c_3 x^2] \cancel{e^{0x}}^{\,1} + [c_4 + c_5 x] e^{x/3}$$

$$\boxed{y(x) = c_1 + c_2 x + c_3 x^2 + c_4 e^{x/3} + c_5 x e^{x/3}}$$

Ex: $y^{(3)} + 3y'' - 4y = 0$

characteristic equation:

$$r^3 + 3r^2 - 4 = 0$$

factor this:

possible roots are factors of -4

$\pm 1, \ \pm 2, \ \pm 4$

Try $r = 1$ and plug it in

$$(1)^3 + 3(1)^2 - 4 = 1 + 3 - 4 = 0 \checkmark$$

$r = 1$ is a root

factor $(r-1)(\qquad\qquad) = 0$

Find this term by polynomial division

$$\begin{array}{r|l} & r^2 + 4r + 4 \\ \hline r-1 & r^3 + 3r^2 + 0r - 4 \\ & -r^3 + r^2 \\ \hline & \qquad 4r^2 \quad 0r \quad -4 \end{array} \checkmark$$

$$
\begin{array}{r}
-r^3 + r^2 \\
\hline
4r^2 + 0r - 4 \\
-4r^2 + 4r \\
\hline
4r - 4 \\
-4r + 4 \\
\hline
0
\end{array}
$$

Factor our characteristic eqn:

$$r^3 + 3r^2 - 4 = (r-1)(r^2 + 4r + 4) = 0$$

$$(r-1)(r+2)^2 = 0$$

Roots: $r = 1$ multiplicity 1

$r = -2$ multiplicity 2

General solution

$$\boxed{y(x) = c_1 e^{x} + c_2 e^{-2x} + c_3 x e^{-2x}}$$

Next Time: Case 3 — complex-valued roots