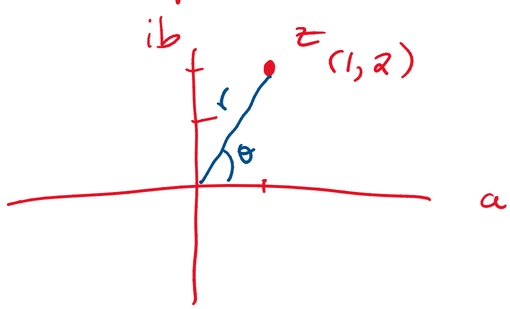
Section 3.3 - Part 2

☆ Homogeneous Equations w/ Constant Coefficients

Warm up: complex numbers

Convert the complex number $z = 1+2i$ into polar form $(z = re^{i\theta})$



$r = \sqrt{a^2+b^2} = \sqrt{1^2+2^2} = \sqrt{5}$

$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{2}{1}\right) \approx 1.11$ rad

$$\boxed{z = \sqrt{5}\, e^{i(1.11)}}$$

$z = a+ib$

I. 2nd Order Linear:

$$ay'' + by' + cy = 0$$

2nd order, linear, constant coeff, homogeneous

1. Find the characteristic equation:

$$ar^2 + br + c = 0$$

2. Find the roots: quadratic formula

$$r_1, r_2 = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

Three cases for the solution

case	roots	general soln
$b^2-4ac > 0$	r_1 and r_2 real & distinct	$y = C_1e^{r_1x} + C_2e^{r_2x}$
$b^2-4ac = 0$	$r = \frac{-b}{2a}$ real, repeated multiplicity $k=2$	$y = C_1e^{rx} + C_2xe^{rx}$

$b^2-4ac<0$	complex-valued $r=\lambda+i\mu$	?

Ex: $y'' + 6y' + 13y = 0$

1. Char. eqn: $r^2+6r+13=0$

2. roots: $r = \frac{-6}{2\cdot 1} \pm \frac{\sqrt{6^2-4\cdot 1\cdot 13}}{2\cdot 1}$

$= -3 \pm \frac{1}{2}\sqrt{36-52}$

$= -3 \pm \frac{1}{2}\sqrt{-16}$

$\boxed{r = -3 \pm 2i}$

3. Find (real-valued) linearly independent solutions

Could choose: $\hat{y}_1 = e^{(-3+2i)x}$

$\hat{y}_2 = e^{(-3-2i)x}$

linearly independent, but <u>not</u> real-valued

<u>Q</u>: Can we find a linear combination of $\hat{y}_1$ and $\hat{y}_2$ so that the solns are real-valued?

<u>Euler's</u> <u>Formula</u>: $e^{i\theta} = \cos\theta + i\sin\theta$

plug in to our solutions

$\hat{y}_1 = e^{-3x}e^{2ix} = e^{-3x}[\cos(2x) + i\sin(2x)]$

$\hat{y}_2 = e^{-3x}e^{-2ix} = e^{-3x}[\cos(-2x) + i\sin(-2x)]$

$\cos(-\theta) = \cos\theta$
$\sin(-\theta) = -\sin(\theta)$

$$\hat{y}_2 = e^{-3x} e^{-2ix} = e^{-3x}\left[\cos(-2x) + i\sin(-2x)\right] \qquad \sin(-\theta) = -\sin(\theta)$$

$$= e^{-3x}\left[\cos(2x) - i\sin(2x)\right]$$

Find a new basis:

$$y_1 = \frac{1}{2}\hat{y}_1 + \frac{1}{2}\hat{y}_2 = \frac{1}{2}\Big[e^{-3x}\big(\cos(2x) + \cancel{i\sin(2x)}\big) + e^{-3x}\big(\cos(2x) - \cancel{i\sin(2x)}\big)\Big]$$

$$= \frac{1}{2}e^{-3x}\big(2\cos(2x)\big)$$

$$\boxed{y_1 = e^{-3x}\cos(2x)} \quad \text{real-valued}$$

$$y_2 = \frac{1}{2i}\hat{y}_1 - \frac{1}{2i}\hat{y}_2 = \frac{1}{2i}\Big[e^{-3x}\big(\cancel{\cos(2x)} + i\sin(2x)\big) + e^{-3x}\big(\cancel{-\cos(2x)} + i\sin(2x)\big)\Big]$$

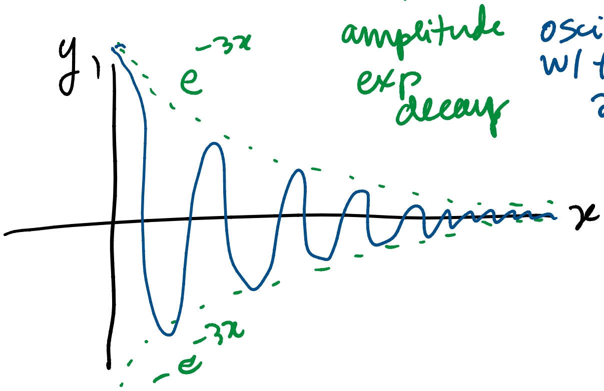
$$= \frac{1}{\cancel{2i}}\left[e^{-3x}\,(\cancel{2i})\sin(2x)\right]$$

$$\boxed{y_2 = e^{-3x}\sin(2x)} \quad \text{real-valued.}$$

y_1 and y_2 real-valued, linearly independent, form a basis for the solutions of the ODE.

General solution:

$$y(x) = \underbrace{C_1 e^{-3x}}_{\substack{\text{amplitude}\\ \text{exp decay}}}\underbrace{\cos(2x)}_{\substack{\text{oscillate}\\ \text{w/ freq } 2}} + C_2 e^{-3x}\sin(2x)$$



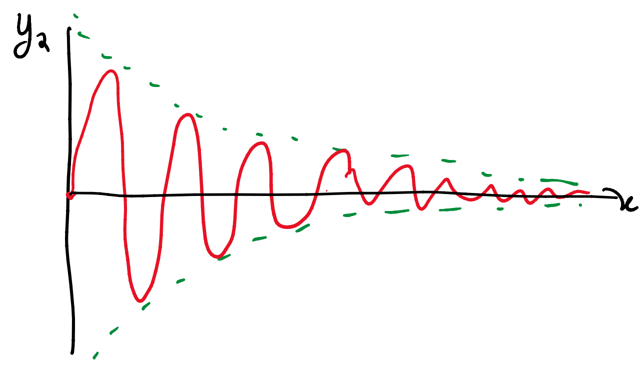


Table:

case	roots	general soln
$b^2-4ac<0$	complex-valued $r=\lambda \pm i\mu$	$y = c_1 e^{\lambda x}\cos(\mu x) + c_2 e^{\lambda x}\sin(\mu x)$

$$r = \frac{-b}{2a} \pm i\frac{\sqrt{4ac-b^2}}{2a} = \lambda \pm i\mu$$

$\lambda = \frac{-b}{2a}$ exp growth or decay

$\mu = \frac{\sqrt{4ac-b^2}}{2a}$ frequency of oscillation.

II. Higher Order:

Q: What happens if we have a repeated complex-valued root?

Ex: $y^{(4)} + 8y'' + 16y = 0$

characteristic eqn: $r^4 + 8r^2 + 16 = 0$

factor: $(r^2+4)^2 = 0$

but $r^2+4=0 \longrightarrow r = \pm 2i$

so we have: $(r+2i)^2(r-2i)^2 = 0$

$$\begin{cases} r_1 = 2i & \text{has multiplicity } k=2 \\ r_2 = -2i & \text{has multiplicity } k=2 \end{cases}$$

Solutions:

$(r_1 = 2i)$ $y_1 = \cancel{e^{0x}}\cos(2x) = \cos(2x)$

solutions

$(r_1 = 2i)$ $\quad y_1 = \cancel{e^{0x}} \cos(2x) = \cos(2x)$

$(r_1 = -2i)$ $\quad y_2 = \cancel{e^{0x}} \sin(2x) = \sin(2x)$

and for multiplicity 2, multiply by x

$$y_3 = x\cos(2x) \qquad y_4 = x\sin(2x)$$

General solution:

$$\boxed{y(x) = (C_1 + C_2 x)\cos(2x) + (C_3 + C_4 x)\sin(2x)}$$

Ex: Let's that an ODE has the characteristic eqn:

$$(r-2)(r+3)^4(r-(5+i))(r-(5-i)) = 0$$

NOTE: All complex roots $\begin{cases} r_1 = a+bi \\ r_2 = a-bi \end{cases}$ come in conjugate pairs

Roots:	$r_1 = 2$	multiplicity	$k=1$
	$r_2 = -3$		$k=4$
	$r_3 = 5 \pm i$		$k=1$

General solution:

$$\boxed{\begin{aligned} y(x) = {} & C_1 e^{2x} + (C_2 + C_3 x + C_4 x^2 + C_5 x^3)e^{-3x} \\ & + C_6 e^{5x}\cos(x) + C_7 e^{5x}\sin(x) \end{aligned}}$$

III. Euler Equations:

$$ax^3 y''' + bx^2 y'' + cxy' + dy = 0$$

where a, b, c, d are constants

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we make a substitution

$$v = \ln x$$

$$a\frac{d^3y}{dv^3} + (b-3a)\frac{d^2y}{dv^2} + (c-b+2a)\frac{dy}{dv} + dy = 0$$

Substitution $v = \ln x$

$$\frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{dx} = \frac{dy}{dx}\left(\frac{1}{x}\right) = \frac{1}{x}\frac{dy}{dv}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dv}\right) = \frac{-1}{x^2}\frac{dy}{dv} + \left(\frac{1}{x}\right)\frac{d^2y}{dv^2}\cdot\frac{dv}{dx}$$

$$= \frac{-1}{x^2}\frac{dy}{dv} + \frac{1}{x^2}\frac{d^2y}{dv^2} \quad \ldots\ldots$$

<u>Ex</u>: $x^3y''' - x^2y'' + xy' = 0$ $\qquad a = 1 \quad c = 1$
$\qquad b = -1 \quad d = 0$

Transform:

$$a\frac{d^3y}{dv^3} + (b-3a)\frac{d^2y}{dv^2} + (c-b+2a)\frac{dy}{dv} + d\,y = 0$$

$$\frac{d^3y}{dv^3} + (-1-3\cdot 1)\frac{d^2y}{dv^2} + (1-(-1)+2\cdot 1)\frac{dy}{dv} + 0\cdot y = 0$$

$$\frac{d^3y}{dv^3} - 4\frac{d^2y}{dv^2} + 4\frac{dy}{dv} = 0$$

3rd order, linear, const coeff.

char. eqn: $r^3 - 4r^2 + 4r = 0$

factor: $r(r^2 - 4r + 4) = 0$

$$r(r-2)^2 = 0$$

Roots: $r = 0$ mult. $k = 1$

$r = 2$ mult $k = 2$

General solution:

$$y(v) = C_1 + C_2 e^{2v} + C_3 v e^{2v}$$

Transform back $v = \ln x$

$$y(x) = C_1 + C_2 e^{2\ln x} + C_3 \ln x \, e^{2\ln x}$$

$$\boxed{y(x) = C_1 + C_2 x^2 + C_3 x^2 \ln x}$$