★ Mechanical Vibrations

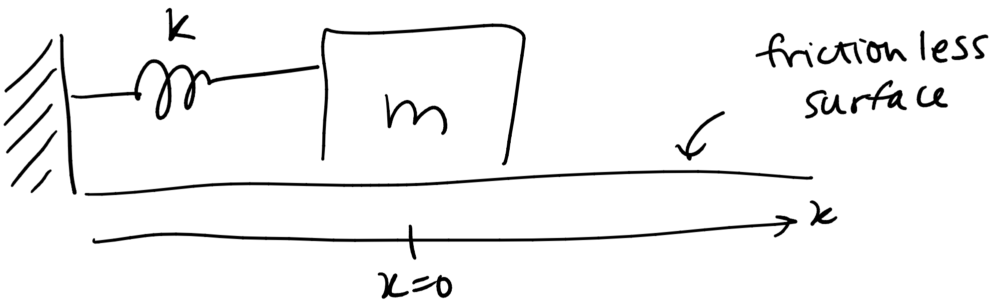
Warm up: The ODE: $x'' + 4x = 0$

Has characteristic equation $r^2 + 4 = 0$
and roots $r = \pm 2i$

Write down the general solution.

Ans: $x(t) = C_1 \cos(2t) + C_2 \sin(2t)$

I. Undamped Mass on a Spring



k - spring constant

m - mass

$x(t)$ - displacement

$x = 0$ rest position of the mass (no force)

Force of the spring on mass

$$F_{spring} = -kx$$

Assume $k > 0$, $m > 0$

Newton's 2nd law:

$$\begin{cases} m x'' = -kx \\ \underbrace{x(0) = x_0}_{\text{initial position}}, \quad \underbrace{x'(0) = v_0}_{\text{initial velocity}} \end{cases}$$

2nd order linear homogeneous constant coefficients

Rewrite: $x'' + \frac{k}{m} x = 0$

Define: $w_0 = \sqrt{\frac{k}{m}}$ natural frequency

characteristic eqn: $r^2 + w_0^2 = 0$

Characteristic eqn: $r^2 + \omega_0^2 = 0$

Roots: $r = \pm i\omega_0$

General solution: $x(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$

Initial Conditions:

$$x(0) = x_0 = \left[C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) \right] \Big|_{t=0}$$

$$x_0 = C_1$$

$$x'(0) = v_0 = \left[-\omega_0 C_1 \sin(\omega_0 t) + \omega_0 C_2 \cos(\omega_0 t) \right] \Big|_{t=0}$$

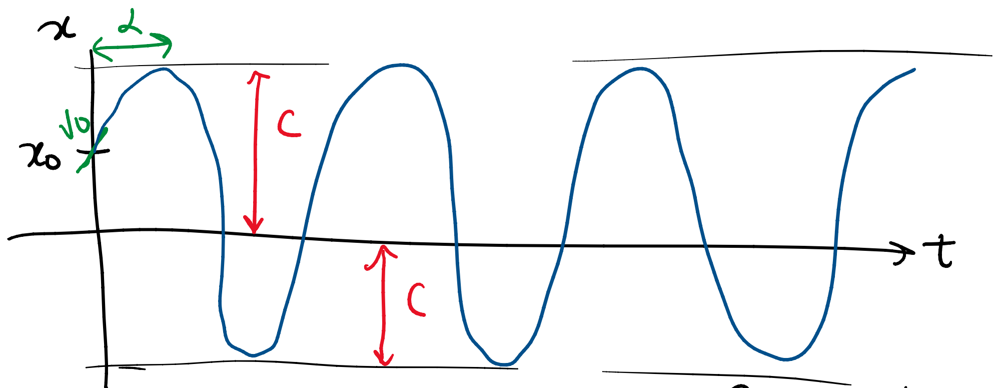
$$v_0 = \omega_0 C_2 \rightarrow C_2 = \frac{v_0}{\omega_0}$$

Solution to IVP is:

$$\boxed{x(t) = x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t)}$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

Plot:



sinusoid with frequency ω_0

In general, we can write

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

in condensed form:

$$x(t) = C\cos(\omega_0 t - \alpha)$$

Terminology: C – amplitude of oscillation [m]

... angular freq... [rad/s]

Terminology: C – amplitude of oscillation [m]
ω_0 – natural freq / circular freq [rad/s]
α – phase angle. [rad]

Formulas: $C = \sqrt{A^2 + B^2}$, $\alpha = \tan^{-1}\left(\frac{B}{A}\right)$

We can simpelify further:

$$x(t) = C\cos(\omega_0(t - \delta))$$

where $\delta = \frac{\alpha}{\omega_0}$ is the time lag [s]

$T = \frac{2\pi}{\omega_0}$ period of oscillation [s]

$\nu = \frac{1}{T} = \frac{\omega_0}{2\pi}$ frequency of oscillation [Hz] = [s^{-1}]

Ex: Undamped spring

$m = \frac{1}{2}$ kg

spring is stretched by 2m by a force of 100N

set in motion with $x_0 = 1$m, $v_0 = -5$ m/s

Find:
- position $x(t)$
- amplitude C
- frequency ν
- period T
- time lag of motion

$x(t) = C\cos(\omega_0(t - \delta))$

spring constant $k = \frac{100N}{2m} = 50$ N/m

Newton's 2nd Law: $\frac{1}{2}x'' = -50x$

IVP: $\begin{cases} x'' + 100x = 0 \\ x(0) = 1 \quad x'(0) = -5 \end{cases}$

IVP: $\begin{cases} x \\ x(0) = 1 \qquad x'(0) = -5 \end{cases}$

natural frequency: $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{100} = 10 \text{ rad/s}$

period: $T = \frac{2\pi}{\omega_0} = \frac{2\pi}{10} \approx 0.6283 \text{ s}$

frequency: $\nu = \frac{\omega_0}{2\pi} = \frac{10}{2\pi} \approx 1.5915 \text{ Hz}$

position: $x(t) = x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t)$

$$= \cos(10t) + \frac{(-5)}{10} \sin(10t)$$

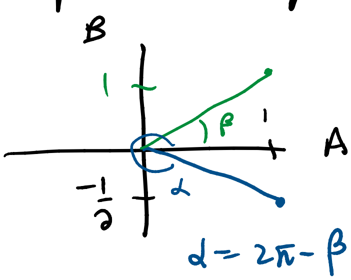
$$x(t) = \cos(10t) - \frac{1}{2} \sin(10t)$$

Rewrite as: $x(t) = C \cos(10(t - \delta))$

$$C = \sqrt{A^2 + B^2} = \sqrt{(1)^2 + \left(\tfrac{1}{2}\right)^2} = \sqrt{\tfrac{5}{4}} = \frac{\sqrt{5}}{2}$$

Amplitude: $C = \frac{\sqrt{5}}{2} \text{ m}$

phase angle: $\alpha = \tan^{-1}\left(\frac{B}{A}\right) = \tan^{-1}\left(\frac{-\frac{1}{2}}{1}\right)$



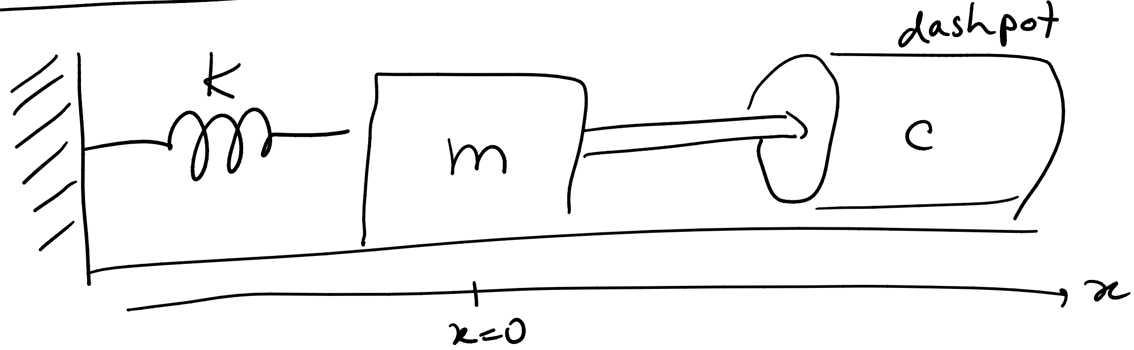
$$= 2\pi - \tan^{-1}\left(\frac{1}{2}\right)$$

$$\alpha \approx 5.8195$$

time lag: $\delta = \frac{\alpha}{\omega_0} = \frac{\alpha}{10} \approx 0.5820$

$$\boxed{x(t) = \frac{1}{2}\sqrt{5} \cos(10(t - 0.5820))}$$

II. Damped Mass on a Spring



mass-spring-dashpot

Dashpot acts opposite the velocity of the mass

$$F_d = -cx'$$

Newton's 2nd law:

$$mx'' = -cx' - kx$$

IVP: $\begin{cases} mx'' + cx' + kx = 0 \\ x(0) = x_0 \quad x'(0) = V_0 \end{cases}$

Assume
$m > 0$
$k > 0$
$c > 0$

Rewrite ODE: $x'' + \frac{c}{m}x' + \frac{k}{m}x = 0$

characteristic eqn: $r^2 + \frac{c}{m}r + \frac{k}{m} = 0$

Roots: quadratic formula

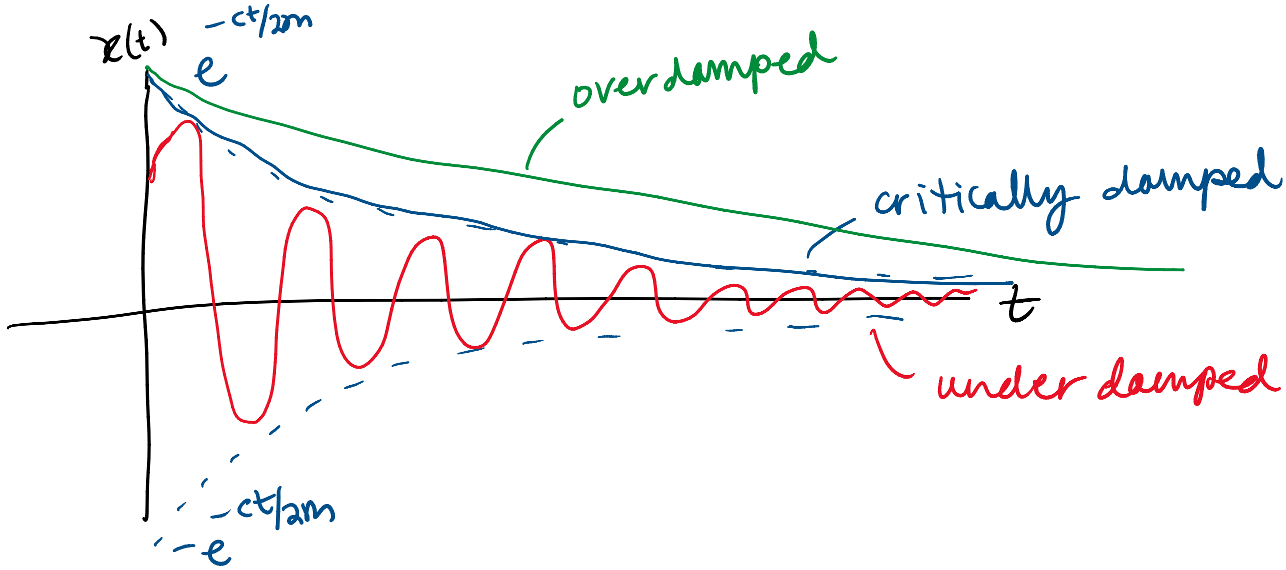
$$r = \frac{\left(-\frac{c}{m}\right)}{2\cdot 1} \pm \frac{\sqrt{\left(\frac{c}{m}\right)^2 - 4\cdot 1\cdot\left(\frac{k}{m}\right)}}{2\cdot 1}$$

$$\boxed{r = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km}}$$

3 cases for solutions

Case 1	Case 2	Case 3
$c^2 - 4km > 0$	$c^2 - 4km = 0$	$c^2 - 4km < 0$

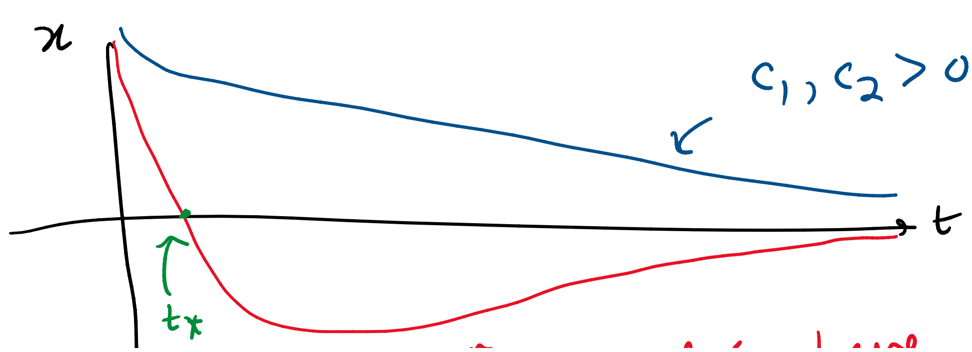
$c^2 - 4km > 0$	$c^2 - 4km = 0$	$c^2 - 4km < 0$
r_1, r_2 real distinct	r real repeated	complex-valued $r = \lambda \pm i\mu$
$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$	$x(t) = c_1 e^{rt} + c_2 t e^{rt}$	$x(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t)$
$r_1 < -\frac{c}{2m} < r_2$		
over damped	critically damped	under-damped

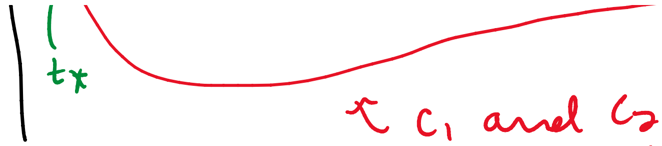


NOTE: In the criticaly damped case:

$$r = -\frac{c}{2m}$$

$$x(t) = (c_1 + c_2 t)e^{-ct/2m}$$





c_1 and c_2 have opposite signs

solution crosses t-axis when

$$(c_1 + c_2 t_*) = 0$$

$$t_* = \frac{-c_1}{c_2}$$