✩ Nonhomogeneous Equations & Undetermined Coefficients

Warm up: Show that $y(t) = 3t + 6$ solves the non homogeneous ODE:

$$y'' - 2y' + y = 3t$$

Ans: plug into the ODE:

$y = 3t+6, \quad y' = 3 \quad y'' = 0$

$$[0] - 2[3] + [3t+6] \stackrel{?}{=} 3t$$

$$-\cancel{6} + 3t + \cancel{6} = 3t \checkmark$$

I. Non-homogeneous ODE:

Linear:

$$(*) \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = f(t)$$

Here $a_n, \dots a_0$ are constant coefficients
we call $f(t)$ the forcing

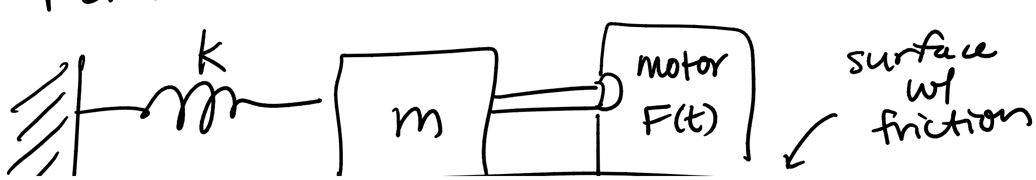
GOAL: Find a particular solution $y_p(t)$ that Solve the non homogeneous eqn $(*)$

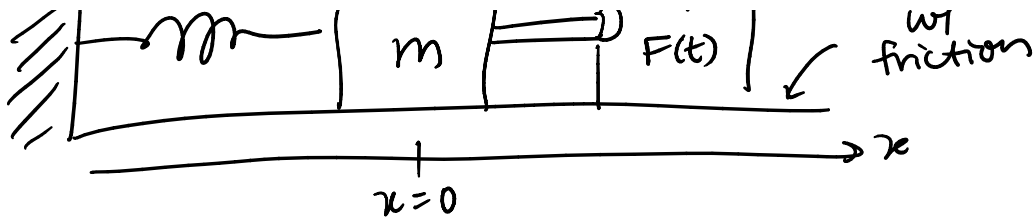
If the general solution $y_c(t)$ (complementary solution) solves the homogeneous eqn ($f(t) \equiv 0$)

Then the full solution

$$y = y_c + y_p$$

Ex: Forced mass on a spring





Then the equations of motion

$$\text{IVP} \quad \begin{cases} x'' + \frac{c}{m}x' + \frac{k}{m}x = \frac{F(t)}{m} \equiv f(t) \\ x(0) = x_0 \ , \quad x'(0) = v_0 \end{cases}$$

Ex: $x'' - 4x = e^{5t}$

1. Find the general soln to $x'' - 4x = 0$

char eqn: $r^2 - 4 = 0$

$(r-2)(r+2) = 0$

Roots: $r = +2, -2$

general solution $x_c(t) = C_1 e^{2t} + C_2 e^{-2t}$

2. Find the particular solution

Method of Undetermined Coefficients

Guess the form of the solution

$x'' - 4x = e^{5t}$

Ansatz: $x_p(t) = A e^{5t}$

Here A is unknown.

We will make Table of Ansatz later

plug x_p into the ODE

$x_p' = 5Ae^{5t}$ $\qquad x_p'' = 25Ae^{5t}$

Then $[25Ae^{5t}] - 4[Ae^{5t}] = e^{5t}$

$[25A - 4A]e^{5t} = e^{5t}$

$= 1$

$$[25A - 4A]e \quad - \quad$$

$$21A = 1$$

$$A = 1/21$$

so particular solution $x_p(t) = \frac{1}{21} e^{5t}$

3. Full solution $x = x_c + x_p$

$$\boxed{x(t) = C_1 e^{2t} + C_2 e^{-2t} + \frac{1}{21} e^{5t}}$$

4. May need solve for the initial conditions

Method of Undetermined Coefficients

$f(t)$	Ansatz $y_p(t)$
e^{kt}	Ae^{kt}
$\cos(wt)$ or $\sin(wt)$	$A\cos(wt) + B\sin(wt)$
t^n	$A_n t^n + A_{n-1} t^{n-1} + \dots + A_1 t + A_0$
$e^{kt}\cos(wt)$ or $e^{kt}\sin(wt)$	$Ae^{kt}\cos(wt) + Be^{kt}\sin(wt)$

Let's do some examples:

Ex: $3y'' + y' - 2y = 2\cos(t)$

Ansatz: $y_p(t) = A\cos(t) + B\sin(t)$

take derivatives!

take derivatives!

$$y_p' = -A\sin(t) + B\cos(t)$$

$$y_p'' = -A\cos(t) - B\sin(t)$$

plug into ODE:

$$3[-A\cos(t) - B\sin(t)] + [-A\sin(t) + B\cos(t)] - 2[A\cos(t) + B\sin(t)] = 2\cos(t)$$

collect like terms

$$[-3A + B - 2A]\cos(t) + [-3B - A - 2B]\sin(t) = 2\cos(t)$$

$$-5A + B = 2 \qquad\qquad -5B - A = 0$$

$$A = -5B$$

$$-5[-5B] + B = 2$$

$$26B = 2$$

$$B = \frac{1}{13} \longrightarrow A = \frac{-5}{13}$$

so the particular soln:

$$\boxed{y_p(t) = -\frac{5}{13}\cos(t) + \frac{1}{13}\sin(t)}$$

Ex: $y'' + 3y' + 4y = 3t + 2$

Ansatz: $y_p(t) = At + B$

take derivs: $y_p' = A \qquad y_p'' = 0$

plug into ODE

$$[0] + 3[A] + 4[At + B] = 3t + 2$$

$[0] + 3[A] + 4[At + B] = 3t + 2$

Collect like terms

$4At + [3A + 4B] = 3t + 2$

$4A = 3$

$A = \frac{3}{4}$

$3A + 4B = 2$

$3\left(\frac{3}{4}\right) + 4B = 2$

$4B = \frac{8}{4} - \frac{9}{4} = \frac{-1}{4}$

$B = -\frac{1}{16}$

particular solution

$$\boxed{y_p(t) = \frac{3}{4}t - \frac{1}{16}}$$

However, this system might not always works.

<u>Counter example</u>: $y'' + 5y' + 6y = 2e^{-3t}$

<u>Ansatz</u>: $y_p(t) = Ae^{-3t}$

Derivs: $y_p' = -3Ae^{-3t}$ $\quad y_p'' = 9Ae^{-3t}$

plug into ODE:

$[9Ae^{-3t}] + 5[-3Ae^{-3t}] + 6[Ae^{-3t}] = 2e^{-3t}$

$$\underbrace{[9A - 15A + 6A]}_{=0}e^{-3t} = 2e^{-3t}$$

$0 \cancel{=} 2e^{-3t}$

$\cancel{\Rightarrow}$

we can't solve for A here

we can't solve for A here

Why not?
→ Look at the general soln

$$y'' + 5y' + 6y = 0$$

char. eqn: $r^2 + 5r + 6 = 0$

$$(r+3)(r+2) = 0$$

roots: $r = -3, -2$

general soln: $y_c(t) = \underbrace{C_1 e^{-3t}}_{\text{this was our Ansatz}} + C_2 e^{-2t}$

e^{-3t} solves the homog. eqn
so it can't. Also solve the non-homog eqn.

So instead, let's try

Ansatz: $y_p(t) = Ate^{-3t}$

derivs: $y_p' = Ae^{-3t} + (At)(-3e^{-3t})$

$$= [A - 3At]e^{-3t}$$

$$y_p'' = -3Ae^{-3t} + [A - 3At](-3e^{-3t})$$

$$= [-6A + 9At]e^{-3t}$$

plug into the ODE:

$$y'' + 5y' + 6y = 2e^{-3t}$$

$$[-6A + 9At]e^{-3t} + 5[A - 3At]e^{-3t} + 6Ate^{-3t} = 2e^{-3t}$$

$[-6A + 9At]e^{-3t} + 5[A - 3At]e^{-3t} + 6At \ldots$

collect like terms

$$[\underbrace{9A - 15A + 6A}_{=0}]te^{-3t} + [-6A + 5A]e^{-3t} = 2e^{-3t}$$

$$-Ae^{-3t} = 2e^{-3t}$$

$$A = -2$$

particular soln: $y_p(t) = -2te^{-3t}$

full solution: $\boxed{y(t) = C_1 e^{-3t} + C_2 e^{-2t} - 2te^{-3t}}$

NOTE: If $f(t)$ appears in the general solution,

choose Ansatz: $y_p(t) = At^s f(t)$

where s is the smallest integer so that no term in $y_p(t)$ duplicates a term $y_c(t)$

New Table:

$f(t)$	$y_p(t)$
e^{kt}	$t^s e^{kt}$
$\cos(\omega t)$ or $\sin(\omega t)$	$t^s[A\cos(\omega t) + B\sin(\omega t)]$
$e^{kt}\cos(\omega t)$ or $e^{kt}\sin(\omega t)$	$t^s[Ae^{kt}\cos(\omega t) + Be^{kt}\sin(\omega t)]$
t^n	$t^s[A_n t^n + \cdots + A_1 t + A_0]$

$t^n e^{kt}$	$t^s\left[A_n t^n + \dots + A_0\right]e^{kt}$
$t^n \cos(\omega t)$ or $t^n \sin(\omega t)$	$t^s\left[(A_n t^n + \dots + A_0)\cos(\omega t) + (B_n t^n + \dots + B_0)\sin(\omega t)\right]$

<u>Ex</u>: $y'' - 2y' + 1 = e^t$

1. find the general solution

char. eqn: $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

root $r = 1$ multiplity $k=2$

$$y_c(t) = C_1 e^t + C_2 t e^t$$

2. find particular solution:

Ansatz: $y_p(t) = A t^s e^t$

want s as small as possible so that there are no duplicate terms in y_c

if $s=1$ $\quad y_p = A t e^t$ ✗

if $s=2$ $\quad y_p(t) = A t^2 e^t$ ✓

so $\boxed{y_p(t) = A t^2 e^t}$

<u>Ex</u>: Set up the form $y_p(t)$ but don't solve for

Ex: Set up the form $y_p(t)$ but don't solve for the coefficients.

$$y^{(4)} + 2y'' + y = t^2\cos(t)$$

1. Find the general soln:

$$y^{(4)} + 2y'' + y = 0$$

char eqn: $r^4 + 2r^2 + 1 = 0$

$$(r^2+1)^2 = 0$$

$$r = \pm i \quad \text{multiplicity } k=2$$

$$y_c(t) = (C_1 + C_2 t)\cos(t) + (C_3 + C_4 t)\sin(t)$$

2. Ansatz: $y_p(t) = t^s\left[(At^2+Bt+C)\cos(t) + (Dt^2+Et+F)\sin(t)\right]$

Need $s=2$

$$y_p(t) = (At^4 + Bt^3 + Ct^2)\cos(t) + (Dt^4 + Et^3 + Ft^2)\sin(t)$$