☆ Variation of Parameters:

Warm up: Consider the nonhomogeneous ODE:

$$y'' + 4y = t^2$$

Suggest an Ansatz to use for the Method of Undetermined Coefficients.

Ansatz: $y_p(t) = At^2 + Bt + C$

I. Computer Project #1:

(Section 3.4 mass-on-a-spring)

Nonlinear spring:

Assume $F_s = kx + \underbrace{\varepsilon x^3}_{\text{nonlinear}}$

Solve the ODE numerically + plot solutions for values of ε.

Submissions:

- MATLAB:
  - ode45 — solves the ODE numerically
  - plot — visualize solutions
- Other computer languages
  - find a numerical ODE solver package
  - plot your solutions
- There are 3 questions:
  - plot solution
  - describe what happens (in words) (hand written, typed, ...)
- Submit one document to Gradescope:
  - code
  - plots
  - written answers.

MATLAB:

IVP: $\begin{cases} x'' + \frac{c}{m}x' + \frac{k}{m}x = 0 \\ x(0) = x_0, \quad x'(0) = V_0 \end{cases}$ $\qquad x'' = -\frac{c}{m}x' - \frac{k}{m}x$

GOAL: use ode45 to solve numerically:

1. Convert to a system of ODE:

Define: $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix}$

where $u_1(t) = x(t)$ $\quad u_2(t) = x'(t)$

Then: $u'(t) = \begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} x'(t) \\ x''(t) \end{bmatrix} = \begin{bmatrix} x'(t) \\ -\frac{c}{m}x' - \frac{k}{m}x \end{bmatrix}$

$$\boxed{u'(t) = \begin{bmatrix} u_2 \\ -\frac{c}{m}u_2 - \frac{k}{m}u_1 \end{bmatrix}}$$

1st order system of ODE

2. Turn it into a function:

```
function dudt = damped_spring(t,u,m,c,k)
    dudt = [u(2); (-c/m)*u(2) - (k/m)*u(1)];
end
```

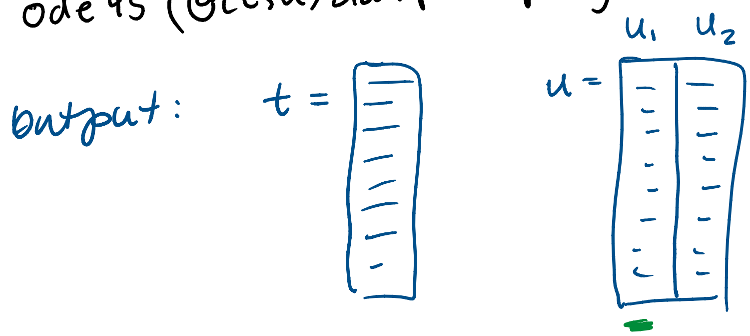
3.
```
// Declare variables:
m = 0.5;
k = 50;
c = 1;
x0 = 1;
V0 = -5;
u0 = [x0; V0];
t_span = [0, 1.5];
```

initial time = 0
end time = 1.5

```
// call on ode45
[t, u] = ode45(@(t,u) damped_spring(t,u,m,c,k), t_span, u0);
```

// call on ode45

[t, u] = ode45(@(t,u) damped_spring(t,u,m,c,k), t_span, u_0);



plot(t, u(:,1));

II. Variation of Parameters:

The Method of Undetermined Coeff doesn't work for all situations.

However, an ODE of the form:

$$(*) \quad 1\, y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \dots + p_1(x)\, y' + p_0(x)\, y = f(x)$$

n-th order, linear, variable coeff, nonhomog

first coeff is 1.

With the general soln:

$$y_c = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

Then $(*)$ has a particular solution that can always be found by Variation of Parameters

Basic Idea:

Ansatz: $y_p(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x) + \dots + u_n(x)\, y_n(x)$

plug into ODE and solve: $u_1(x), \dots, u_n(x)$.

Case $n=2$: $\quad y'' + P(x)\, y' + Q(x)\, y = f(x)$

Case $n=2$: $\quad y'' + P(x)y' + Q(x)y = 0$

with general soln:

$$y_c(x) = C_1 y_1(x) + C_2 y_2(x)$$

Variation of Parameters:

Ansatz: $\quad y_p(x) = u_1(x)\, y_1 + u_2(x)\, y_2$

Take Derivs:

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

$$= (u_1 y_1' + u_2 y_2') + (u_1' y_1 + u_2' y_2)$$

Assume: $\boxed{u_1' y_1 + u_2' y_2 = 0}$

b/c we don't want u_1'' or u_2'' terms

$$y_p' = u_1 y_1' + u_2 y_2'$$

2nd Deriv:

$$y_p'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

$$= (u_1 y_1'' + u_2 y_2'') + (u_1' y_1' + u_2' y_2')$$

But y_1 and y_2 solve the homog. ODE:

$$y_1'' + P(x) y_1' + Q(x) y_1 = 0$$

$$y_2'' + P(x) y_2' + Q(x) y_2 = 0$$

Rewrite

$$\begin{cases} y_1'' = -P(x) y_1' - Q(x) y_1 \\ y_2'' = -P(x) y_2' - Q(x) y_2 \end{cases}$$

$$y_p'' = u_1\left[-P(x) y_1' - Q(x) y_1\right] + u_2\left[-P(x) y_2' - Q(x) y_2\right] + u_1' y_1' + u_2' y_2'$$

$$= (u_1' y_1' + u_2' y_2')$$
$$- P(x)\big[\underbrace{u_1 y_1' + u_2 y_2'}_{y_p'(x)}\big]$$
$$- Q(x)\big[\underbrace{u_1 y_1 + u_2 y_2}_{y_p(x)}\big]$$

Ansatz

$$y_p'' = (u_1' y_1' + u_2' y_2') - P(x)\, y_p' - Q(x)\, y_p$$

$$f(x) = y_p'' + P(x) y_p' + Q(x) y_p = u_1' y_1' + u_2' y_2'$$

$$\boxed{u_1' y_1' + u_2' y_2' = f(x)}$$

Collect equations:

$$(\square) \quad \begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = f(x) \end{cases}$$

system of 2 eqns
2 unknowns $u_1'(x)$, $u_2'(x)$

Write this as a matrix eqn:

$$\underbrace{\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}}_{\text{Wronskian}} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$W(y_1, y_2) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \neq 0$$

b/c y_1 and y_2 are general soln
→ linearly independent

→ this system must have a unique soln.

Variation of Parameters:

1. Solve system $(\square)$ to get u_1' and u_2'

Variation of

1. Solve system ($\square$) to get u_1' and u_2'
2. Integrate $u_1 = \int u_1'\,dx \qquad u_2 = \int u_2'\,dx$
3. $y_p(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$

Ex: $x^2 y'' + xy' - y = 72x^5$

general soln: $y_c(x) = c_1 x + \frac{c_2}{x}$

To use variation of parameters - divide by x^2

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 72x^3$$

2. Take derivs: of y_1 and y_2

$y_1 = x \qquad\qquad y_2 = \frac{1}{x}$

$y_1' = 1 \qquad\qquad y_2' = \frac{-1}{x^2}$

3. Write out system ($\square$)

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = f(x)$$

$$\begin{cases} u_1'(x) + u_2'\left(\frac{1}{x}\right) = 0 \\ u_1'(1) + u_2'\left(\frac{-1}{x^2}\right) = 72x^3 \end{cases}$$

4. Solve for u_1' and u_2'

$$u_1' x + \frac{u_2'}{x} = 0 \longrightarrow u_1' = \frac{-u_2'}{x^2}$$

$$u_1' - \frac{u_2'}{x^2} = 72x^3$$

$$\frac{-u_2'}{x^2} - \frac{u_2'}{x^2} = \frac{-2}{x^2}u_2' = 72x^3$$

$$\rightarrow u_2' = -36x^5$$

$$u_1' = \frac{-u_2}{x^2} = 36x^3$$

5. Integrate to get u_1 and u_2

$$u_1 = \int 36x^3\,dx = \frac{36x^4}{4} = 9x^4$$

$$u_2 = \int -36x^5\,dx = \frac{-36x^6}{6} = -6x^6$$

6. Particular solution:

$$y_p(x) = u_1y_1 + u_2y_2$$

$$= (9x^4)(x) + (-6x^6)\left(\tfrac{1}{x}\right)$$

$$= 9x^5 - 6x^5$$

$$\boxed{y_p(x) = 3x^5}$$

Ex: $y'' + 3y' + 2y = 4e^x$

1. Find the general soln:

char eqn: $r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0 \qquad r = -2, -1$

$$y_c(x) = c_1e^{-x} + c_2e^{-2x}$$

2. Take derivs of y_1 and y_2

$y_1 = e^{-x} \qquad y_2 = e^{-2x}$

$y_1' = -e^{-x} \qquad y_2' = -2e^{-2x}$

3. Write down system $(\square)$

$u_1'y_1 + u_2'y_2 = 0$

$u_1'y_1' + u_2'y_2' = f(x)$

$$\begin{cases} u_1'e^{-x} + u_2'e^{-2x} = 0 \\ u_1'(-e^{-x}) + u_2'(-2e^{-2x}) = 4e^x \end{cases}$$

4. Solve for u_1' and u_2'

$$u_1' e^{-x} + u_2' e^{-2x} = 0$$

$$\rightarrow \underline{u_1'} = -\frac{u_2' e^{-2x}}{e^{-x}} = \underline{-e^{-x} u_2'}$$

$$\underline{-e^{-x} u_1' - 2e^{-2x} u_2' = 4e^{x}}$$

$$-e^{-x}\left(-e^{-x} u_2'\right) - 2e^{-2x} u_2' = 4e^{x}$$

$$-e^{-2x} u_2' = \left(e^{-2x} - 2e^{-2x}\right) u_2' = 4e^{x}$$

$$u_2' = \frac{-4e^{x}}{e^{-2x}} = -4e^{3x}$$

$$u_1' = -e^{-x}\left(-4e^{3x}\right) = 4e^{2x}$$

5. Integrate to get u_1 and u_2

$$u_1 = \int 4e^{2x}\,dx = \frac{4e^{2x}}{2} = 2e^{2x}$$

$$u_2 = \int -4e^{3x}\,dx = -\frac{4e^{3x}}{3} = -\frac{4}{3}e^{3x}$$

6. Particular soln:

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(2e^{2x}\right)\left(e^{-x}\right) + \left(-\frac{4}{3}e^{3x}\right)\left(e^{-2x}\right)$$

$$= 2e^{x} - \frac{4}{3}e^{x}$$

$$\boxed{y_p = \frac{2}{3}e^{x}}$$