Section 3.6

Forced Oscillations & Resonance

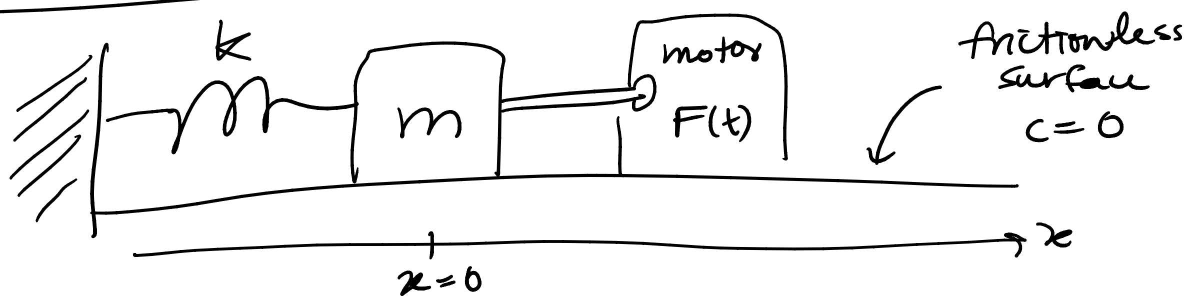
Warm-up: Consider the undamped mass-on-a-spring problem:

$$4x'' + 100x = 0$$

What is the natural frequency ω_0?

Ans: $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{4}} = \sqrt{25} = 5$

I. Undamped Forced Oscillations



Equations of motion:

$$\begin{cases} x'' + \frac{k}{m}x = \frac{F(t)}{m} = f(t) \\ x(0) = x_0, \quad x'(0) = v_0 \end{cases}$$

nonhomog

Recall: natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$

Common forcing terms: sinusoid

$$f(t) = A\cos(\omega t) + B\sin(\omega t)$$

ω — external frequency

Two cases:

case 1: $\omega \neq \omega_0$

case 2: $\omega = \omega_0$

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$$x'' + 9x = 80\cos(5t)$$

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Ex: $\begin{cases} x'' + 9x = 80\cos(5t) \\ x(0) = 0, \quad x'(0) = 0 \end{cases}$

1. Find the general soln: $x'' + 9x = 0$

$\omega_0 = \sqrt{9} = 3$

$x_c(t) = C_1 \cos(3t) + C_2 \sin(3t)$

2. Particular soln - Method of Undetermined Coeff

Ansatz: $x_p(t) = A\cos(5t) + B\sin(5t)$

Take derivs: $x_p' = -5A\sin(5t) + 5B\cos(5t)$

$x_p'' = -25A\cos(5t) - 25B\sin(5t)$

plug into ODE: $x'' + 9x = 80\cos(5t)$

$$[-25A\cos(5t) - 25B\sin(5t)] + 9[A\cos(5t) + B\sin(5t)] = 80\cos(5t)$$

collect like terms:

$$[-25A + 9A]\cos(5t) + [-25B + 9B]\sin(5t) = 80\cos(5t)$$

$-16A = 80$

$A = -5$

$-16B = 0$

$B = 0$

so $x_p(t) = -5\cos(5t)$

3. Full soln: $x = x_c + x_p$

$x(t) = C_1\cos(3t) + C_2\sin(3t) - 5\cos(5t)$

4. Plug in initial conditions:

$x(0) = 0 = C_1 - 5 \longrightarrow C_1 = 5$

$\longrightarrow C_2 = 0$

$$x(0)=0 = c_1 - 5$$
$$x'(0)=0 = 3c_2 \longrightarrow c_2 = 0$$

so the solution:

$$x(t) = 5\cos(3t) - 5\cos(5t)$$

Simplify further using trig identity

$$2\sin(A)\sin(B) = \cos(A-B) - \cos(A+B)$$

to apply: $A - B = 3t$, $A + B = 5t$ $\Longrightarrow$ $A = 4t$, $B = t$

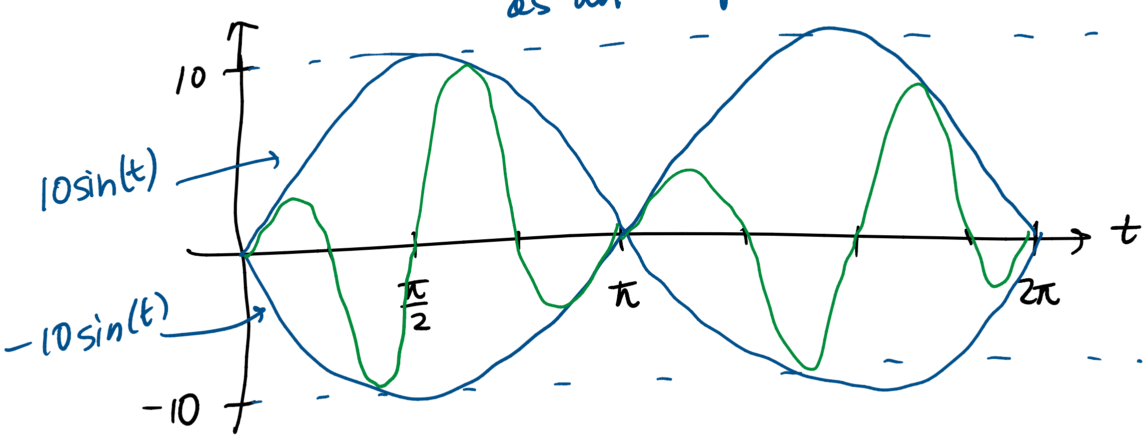
$$x(t) = 5\left[\cos(3t) - \cos(5t)\right]$$
$$= 5\left[2\sin(4t)\sin(t)\right]$$

$$\boxed{x(t) = 10\underbrace{\sin(t)}_{\text{varies slowly}}\underbrace{\sin(4t)}_{\text{varies rapidly}}}$$

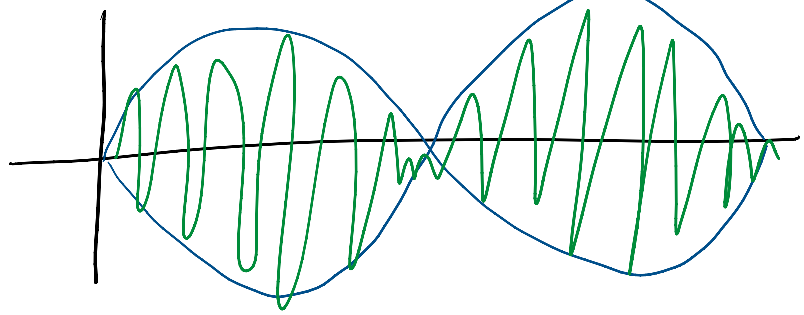
$T = \frac{2\pi}{4} = \frac{\pi}{2}$

varies slowly: think of this as an amplitude

varies rapidly: oscillate 4x as much as



Beats: A rapid oscillation with a (comparatively) slow-varying amplitude slow-varying amplitude



In most general case:

$$\begin{cases} x'' + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t) \\ x(0) = 0, \quad x'(0) = 0 \end{cases}$$

The soln:

$$x(t) = \underbrace{\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{(\omega_0 - \omega)t}{2}\right)}_{\text{amplitude slow-varying}} \underbrace{\sin\left(\frac{(\omega_0 + \omega)t}{2}\right)}_{\text{fast oscillation.}}$$

<u>Case 2:</u> $\omega = \omega_0$

IVP: $\begin{cases} x'' + \omega_0^2 x = \frac{F_0}{m}\cos(\omega_0 t) \\ x(0) = 0, \quad x'(0) = 0 \end{cases}$

<u>Ex:</u> $\begin{cases} x'' + 9x = 60\cos(3t) \\ x(0) = 0, \quad x'(0) = 0 \end{cases}$

1. General soln: $\omega_0 = 3$

$$x_c(t) = C_1\cos(3t) + C_2\sin(3t)$$

2. Find the particular soln:

... undetermined Coeff.

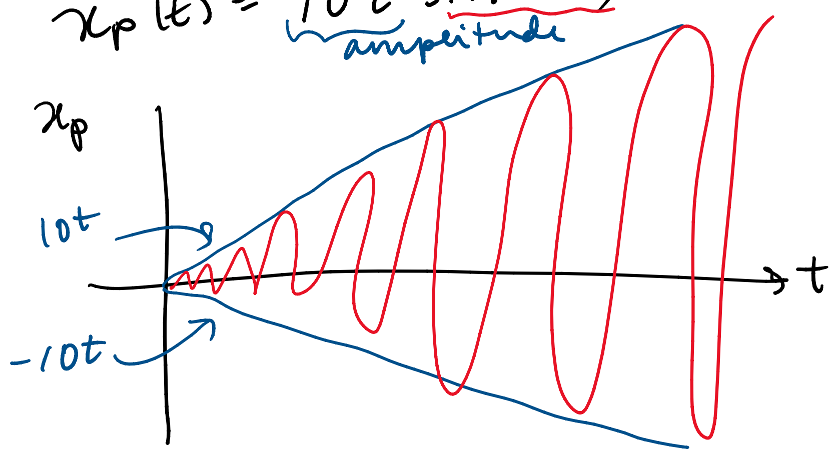
2. Find the particular soln:
Method of undetermined Coeff.

<u>Ansatz</u>: $x_p(t) = t\left[A\cos(3t) + B\sin(3t)\right]$

→ plug into ODE.

$A = 0, \quad B = 10$

$x_p(t) = \underbrace{10t}_{\text{amplitude}} \sin(3t)$



→ amplitude increases without bound when $\omega = \omega_0$ → pure <u>resonance</u>

<u>Ex</u>: Tacoma Bridge disaster

II. Damped Forced Oscillations:

$$mx'' + cx' + kx = F_0\cos(\omega t)$$

1. Find the general soln: $x_c(t)$
   - underdamped, overdamped, critically damped
   - all cases, $x_c(t) \longrightarrow 0$ as $t \longrightarrow 0$
   - $x_c(t)$ is a <u>transient solution</u>

     dies out as $t \to \infty$

steady

dies out as t ...

2. Particular soln $x_p(t)$ $\leftarrow$ steady periodic soln

3. Full soln: $x(t) = x_c + x_p \xrightarrow[t \to \infty]{} x_p(t)$

$\rightarrow$ Find particular soln:

Ansatz: $x_p(t) = A\cos(\omega t) + B\sin(\omega t)$

plug into ODE

$$A = \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + (c\omega)^2} \qquad B = \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2}$$

We want simplify to:

$$x_p(t) = C\cos(\omega t - \alpha)$$

where:

$$C = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

$$\tan\alpha = \frac{c\omega}{k - m\omega^2} \qquad \text{with} \quad 0 < \alpha < \pi$$

$$\alpha = \begin{cases} \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) & \text{if } k > m\omega^2 \\ \pi + \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) & \text{if } k < m\omega^2 \\ \frac{\pi}{2} & \text{if } k = m\omega^2 \end{cases}$$

$$x_p(t) = C\cos(\omega t - \alpha)$$

$$C = \frac{F_0}{} < \infty$$

$$C = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}} < \infty$$

always finite

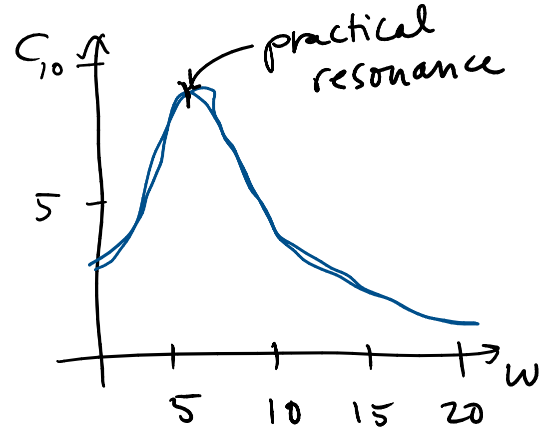
but $C = C(\omega)$ is a function of ω

Def: The practical resonance occurs when $C = C(\omega)$ achieves a maximum value as function of ω.

Ex: consider the case: $m = 1$, $c = 2$, $k = 26$, $F_0 = 82$

$$C = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}} = \frac{82}{\sqrt{(26-\omega^2)^2 + (2\omega)^2}}$$

$$= \frac{82}{\sqrt{676 - 48\omega^2 + \omega^4}}$$



so to find the practical resonance

set $C'(\omega) = 0$

$\frac{dC}{d\omega} = 0$ and solve for ω

$\omega = \sqrt{24} \approx 4.9$