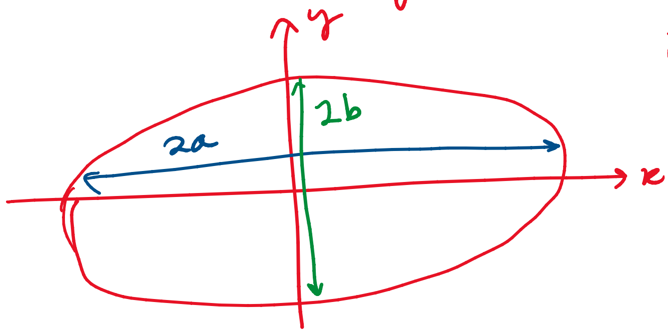
☆ First Order Systems & Applications:

Warm up: What does the plot of the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

in the x-y plane look like?

Ellipse



when $a=b$, then this gives a circle w/ radius a.

I. Systems of Differential Equations

So far, ODEs

$$\begin{cases} \dfrac{dy}{dx} = f(y,x) & \text{1st order} \\ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = f(x) & \text{nth order, linear} \end{cases}$$

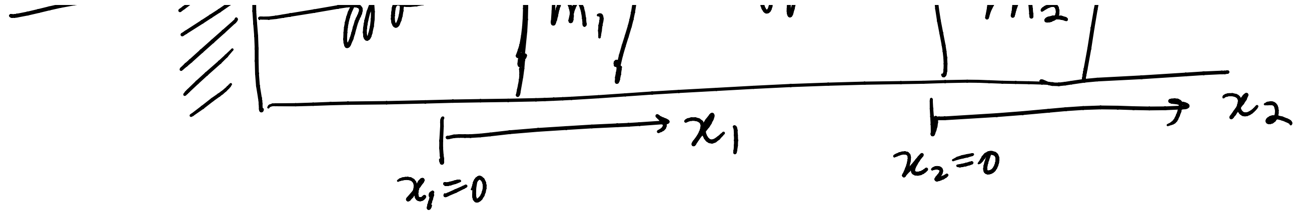
For both, GOAL: find the function $y(x)$

However: many applications have: independent var x

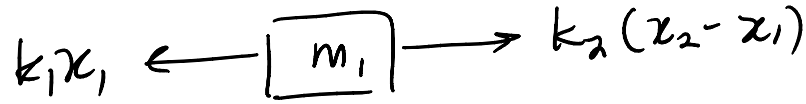
dependent var: $y_1, y_2, \dots, y_n$

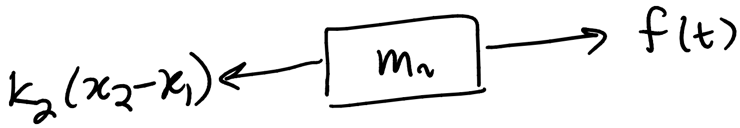
$\Longrightarrow$ system of ODEs

Ex:



we have 2 displacement fcns: $x_1(t)$, $x_2(t)$





Newton's 2nd law:

$$\begin{cases} m_1x_1'' = -k_1x_1 + k_2(x_2-x_1) \\ m_2x_2'' = -k_2(x_2-x_1) + f(t) \end{cases}$$

2nd order system of 2nd order ODE

II. First Order Systems

Any nth order ODE (or system of ODEs) can be transformed into an equivalent 1st order system of ODEs

Ex: $mx'' + cx' + kx = f(t)$

1 ODE, 2nd order

$n = \#$ of ODE $= 1$

$o =$ order $= 2$

total $= n \cdot o = 1 \cdot 2 = 2$

Define: $x_1 = x \qquad x_2 = x'$

then:
$$x_1' = x' = x_2$$
$$x_2' = x'' = \frac{f(t)}{m} - \frac{c}{m}x' - \frac{k}{m}x$$
$$= \frac{f(t)}{m} - \frac{c}{m}x_2 - \frac{k}{m}x_1$$

Equivalent 1st order system:

$$\begin{cases} x_1' = x_2 \\ x_2' = \frac{f(t)}{m} - \frac{c}{m}x_2 - \frac{k}{m}x_1 \end{cases}$$

$n = \#$ of eqns $= 2$ $\quad$ $o =$ order $= 1$

total $= n \cdot o = 2 \cdot 1 = 2$

Ex: $\quad x''' + 3x'' + 2x' - 5x = \sin(2t)$

$n = 1$ $\quad$ $o = 3$ $\quad$ total $= n \cdot o = 3$

$\parallel$

total $= 3$

Transform: $\quad n = 3 \quad o = 1$

Define: $\quad x_1 = x, \quad x_2 = x', \quad x_3 = x''$

Equivalent system
$$\begin{cases} x_1' = x' = x_2 \\ x_2' = x'' = x_3 \\ x_3' = x''' = \sin(2t) - 3x'' - 2x' + 5x \\ \quad = \sin(2t) - 3x_3 - 2x_2 + 5x_1 \end{cases}$$

NOTE: Transforming into a 1st order system is especially useful with nonlinear ODE (because they can be solved numerically)

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Ex: $x'' = x^3 + (x')^3$

$n = 1$ $\quad o = 2$ $\quad$ total $= n \cdot o = 2$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$

Transform: $n = 2$ $\quad o = 1$ $\quad$ total $= 2$

Define: $x_1 = x \qquad x_2 = x'$

$$\begin{cases} x_1' = x' = x_2 \\ x_2' = x'' = x^3 + (x')^3 = x_1^3 + x_2^3 \end{cases}$$

NOTE: This also works for higher order systems of ODE

Ex: $2x'' = -6x + 2y$
$\quad y'' = 2x - 2y + 40\sin(3t)$

$n = 2$ $\quad o = 2$ $\quad$ total $= n \cdot o = 4$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$

Transform: $n = 4$ $\quad o = 1$ $\quad$ total $= 4$

Define: $x_1 = x, \quad x_2 = x', \quad y_1 = y, \quad y_2 = y'$

Equivalent system

$$\begin{cases} x_1' = x' = x_2 \\ x_2' = x'' = \frac{1}{2}(-6x + 2y) = -3x_1 + y_1 \\ y_1' = y' = y_2 \\ y_2' = y'' = 2x - 2y + 40\sin(3t) = 2x_1 - 2y_1 + 40\sin(3t) \end{cases}$$

III. Simple 2D Systems:

$x'' + px' + qx = 0$

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2nd order, linear, homogeneous ODE w/ constant coefficiens

Equivalent 1st order system:

$$\begin{cases} x' = y \\ y' = -qx - py \end{cases}$$

$x_1 = x = x$
$x_2 = x' = y$

Ex: Solve the 2D system:

$$\begin{cases} x' = -2y \\ y' = \frac{1}{2}x \end{cases}$$

Convert back to 1 eqn

$$x'' = -2y' = -2\left(\tfrac{1}{2}x\right) = -x$$

so $x'' + x = 0$

char. eqn: $r^2 + 1 = 0$

Roots: $r = \pm i$

general soln: $x(t) = A\cos(t) + B\sin(t)$

$$\boxed{x(t) = C\cos(t-\alpha)}$$

Then $y(t)$ satisfy $-2y = x'$

$$y = \frac{-1}{2}x' = \frac{-1}{2}\left[-C\sin(t-\alpha)\right]$$

$$\boxed{y(t) = \frac{C}{2}\sin(t-\alpha)}$$

NOTE: From trig, we know that:

$$\cos^2\theta + \sin^2\theta = 1$$

let $\theta = t - \alpha$

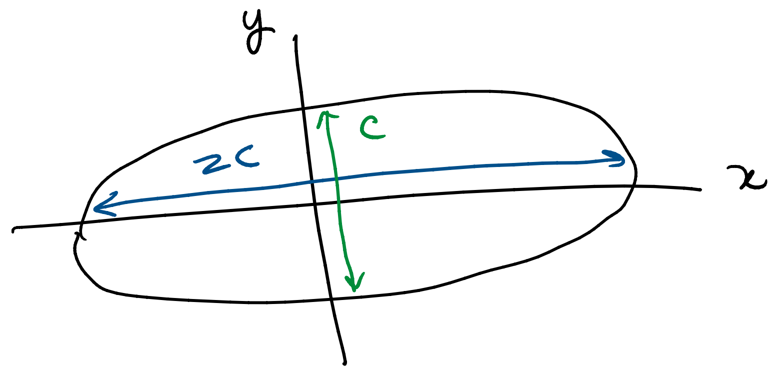
$\cos \theta$ ...

Let $\theta = t - \alpha$

$$\cos^2(t-\alpha) + \sin^2(t-\alpha) = 1$$

$$\left[\frac{x}{c}\right]^2 + \left[\frac{y}{(c/2)}\right]^2 = 1$$

Ellipse! $$\frac{x^2}{c^2} + \frac{y^2}{(c/2)^2} = 1$$



the trajectories are an ellipse

Def: A solution $(x(t), y(t))$ of a 2D system of ODEs

$$\begin{cases} x' = f(t, x, y) \\ y' = g(t, x, y) \end{cases}$$

may be regarded as a parameterization of a solution curve or trajectory of the system in the x-y plane