Section 4.2

☆ Method of Elimination:

Warm up: Convert the ODE:

$$x'' - 7x = 0$$

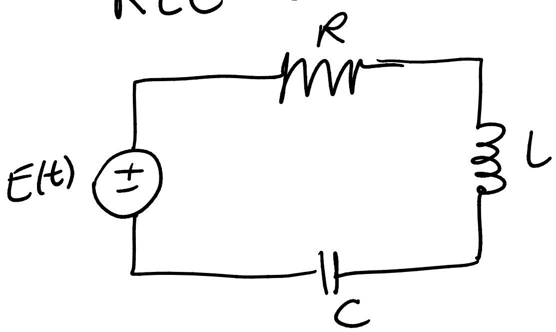
into a 1st order system of ODE

Ans: let $x = x$, $y = x'$

$$\begin{cases} x' = y \\ y' = 7x \end{cases}$$

I. Project #2: due Sat Mar 27 @ 11:59 pm

RLC circut



Goal: to find the charge $Q(t)$ on capacitor

IVP:
$$\begin{cases} LQ'' + RQ' + \frac{1}{C}Q = E(t) = E_0\cos(\omega t) \\ Q(0) = 0, \quad Q'(0) = 0 \end{cases}$$

2nd order, linear, non-homog., w/ const. coeff

Obs: equivalent to Damped Forced Oscillations

$$mx'' + cx' + kx = F_0\cos(\omega t)$$

use knowledge about mass-on-spring problem to interpret solutions of the RLC circuit.

II. Method of Elimination

- 1 approach to solving a system of ODE

#. Method

- 1 approach to solving a system of ODE

GOAL: Reduce the system into a higher order ODE in one variable and solve.

Ex: $\begin{cases} x' = 4x - 3y \\ y' = 6x - 7y \end{cases}$ $\quad x(0) = 2$, $\quad y(0) = -1$

1. Write x in terms of y

$y' = 6x - 7y$ 2nd eqn

$$x = \frac{7}{6}y + \frac{1}{6}y'$$

deriv: $x' = \frac{7}{6}y' + \frac{1}{6}y''$

2. Substitute into the 1st eqn

1st eqn: $x' = 4x - 3y$

$$\left[\frac{7}{6}y' + \frac{1}{6}y''\right] = 4\left[\frac{7}{6}y + \frac{1}{6}y'\right] - 3y$$

simplify

$$\frac{1}{6}y'' + \left[\frac{7}{6} - \frac{4}{6}\right]y' + \left[3 - \frac{4\cdot 7}{6}\right]y = 0$$

$$y'' + 3y' - 10y = 0$$

2nd order, linear, homog, const. coeffs

3. Solve for $y(t)$

char eqn: $r^2 + 3r - 10 = 0$

$$(r+5)(r-2) = 0$$

Roots: $r = 2, -5$

$$y(t) = c_1 e^{2t} + c_2 e^{-5t}$$

4. Find $x(t)$

$$x = \frac{7}{6}y + \frac{1}{6}y'$$

$$= \frac{7}{6}\left[C_1 e^{2t} + C_2 e^{-5t}\right] + \frac{1}{6}\left[2C_1 e^{2t} - 5C_2 e^{-5t}\right]$$

$$x(t) = \frac{3}{2}C_1 e^{2t} + \frac{1}{3}C_2 e^{-5t}$$

5. Plug in the initial conditions

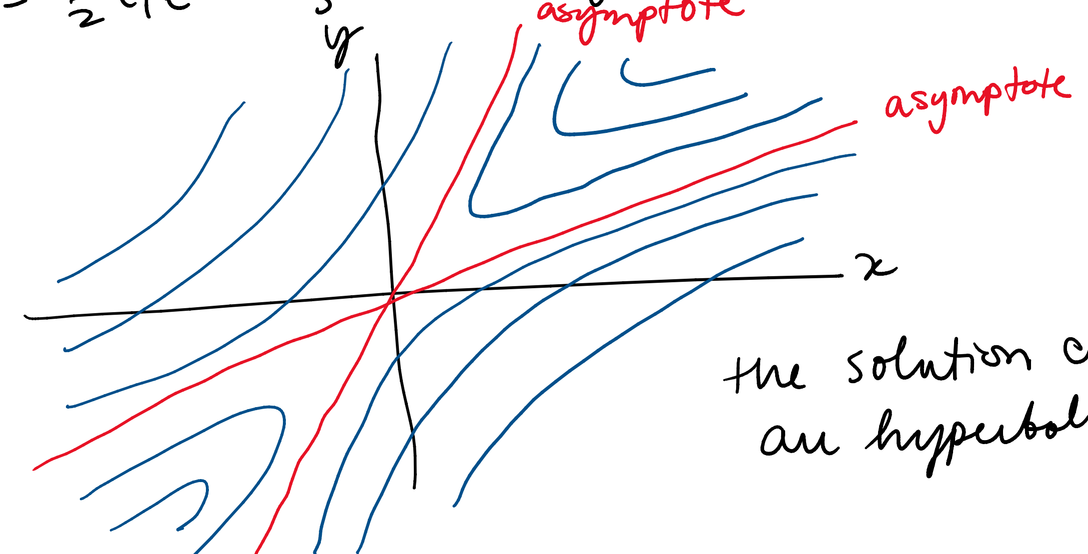
$$\begin{aligned} x(0) &= 2 = \left[\tfrac{3}{2}C_1 + \tfrac{1}{3}C_2\right] \\ y(0) &= -1 = \left[C_1 + C_2\right] \end{aligned} \Rightarrow \begin{aligned} C_1 &= 2 \\ C_2 &= -3 \end{aligned}$$

6. Complete solution:

$$\boxed{\begin{aligned} x(t) &= 3e^{2t} - e^{-5t} \\ y(t) &= 2e^{2t} - 3e^{-5t} \end{aligned}}$$

plot the general solution curves in the x-y plane

$$x(t) = \frac{3}{2}C_1 e^{2t} + \frac{1}{3}C_2 e^{-5t}, \quad y(t) = C_1 e^{2t} + C_2 e^{-5t}$$



the solution curves are hyperbolas

We can write these in vector form:

$$\begin{bmatrix} x \\ y \end{bmatrix} = C_1 \begin{bmatrix} \frac{3}{2}e^{2t} \\ e^{2t} \end{bmatrix} + C_2 \begin{bmatrix} \frac{1}{3}e^{-5t} \\ e^{-5t} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} \frac{3}{2}e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} 3e^{-5t} \\ e^{-5t} \end{bmatrix}$$

solves the system:

$$x' = 4x - 3y$$
$$y' = 6x - 7y$$

vector form:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Def: A system is said to be degenerate if it doesn't have a unique solution. (i.e. it may have no solution or infinitely many solutions)

Ex:

$$2(x' - y' = 0)$$
$$2x' - 2y' = 1$$

$$2x' - \cancel{2y'} = 0$$
$$2x' - \cancel{2y'} = 1$$

write as an eqn in x

$$x' = y' = \frac{1}{2}(-1 + 2x') = -\frac{1}{2} + x'$$

$$0 \neq -\frac{1}{2} \quad \Rightarrow \Leftarrow$$

no solution exists

Degenerate

Ex:

$$2(x' + y' = t)$$
$$2x' + 2y' = 2t$$

$$\begin{cases} 2x' + 2y' = 2t \\ \cancel{2x' + 2y' = 2t} \end{cases}$$

1 eqn, 2 unknowns

write as an eqn in x

$$x' = t - y' = t - [t - x'] = x'$$

tautology

infinitely many solutions

Degenerate

Elimination also works

NOTE: The Method of Elimination also works for nonhomogeneous systems.

Ex: $\begin{cases} x' = 2x + y \\ y' = x + 2y - e^{2t} \end{cases}$ non homog.

1. use 1st eqn to solve for y

$$y = x' - 2x$$

deriv: $y' = x'' - 2x'$

2. plug into 2nd eqn to get ODE in x

$$y' = x + 2y - e^{2t}$$

$$[x'' - 2x'] = x + 2[x' - 2x] - e^{2t}$$

simplify

$$x'' + [-2-2]x' + [-1+4]x = -e^{2t}$$

$$x'' - 4x' + 3x = -e^{2t}$$

2nd order, linear, nonhomog, const. coeff

3. Solve ODE in x

a. find the general soln:

char eqn: $r^2 - 4r + 3 = 0$

$(r-3)(r-1) = 0$

Roots: $r = 1, 3$

$x_c(t) = c_1 e^{t} + c_2 e^{3t}$

b. find particular soln: Ansatz: $x_p(t) = Ae^{2t}$

$$[4Ae^{2t}] - 4[2Ae^{2t}] + 3[Ae^{2t}] = -e^{2t}$$

$$[4A - 8A + 3A]e^{2t} = -e^{2t} \rightarrow A = 1$$

$$[4A - 8A + 3A]e^{2t} = -e^{2t}$$

$$-A = -1 \longrightarrow A = 1$$

$$x_p(t) = e^{2t}$$

full soln: $x(t) = C_1 e^{t} + C_2 e^{3t} + e^{2t}$

4. Find $y(t)$

$$y = x' - 2x$$

$$= [C_1 e^{t} + 3C_2 e^{3t} + 2e^{2t}]$$

$$- 2[C_1 e^{t} + C_2 e^{3t} + e^{2t}]$$

$$= -C_1 e^{t} + C_2 e^{3t}$$

The solution to system:

$$\boxed{\begin{aligned} x(t) &= C_1 e^{t} + C_2 e^{3t} + e^{2t} \\ y(t) &= -C_1 e^{t} + C_2 e^{3t} \end{aligned}}$$