☆ Matrices & Linear Systems

Warm up: Add the two matrices

$\underline{\underline{A}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\underline{\underline{B}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Ans: $\underline{\underline{A}} + \underline{\underline{B}} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

GOAL: use linear algebra to solve systems of ODE

Today: review of Linear Algebra

I. Matrices:

A $m \times n$ matrix $\underline{\underline{A}}$

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

m - rows
n - columns

$\underline{\underline{A}}$ is a matrix $m \times n$

$\underline{x}$ is a vector $(n \times 1)$ $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

a) matrix addition
add terms elementwise

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

b) scalar multiplication

$$\underset{\text{scalar}}{2} \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{\text{matrix}} = \begin{bmatrix} 2\cdot 1 & 2\cdot 2 \\ 2\cdot 3 & 2\cdot 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

mutiply each element by 2

A^T

scalar matrix mutiply

c) transpose of a matrix $\underline{\underline{A}}^T$ row $\rightarrow$ column

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad (2\times 2) \qquad \underline{\underline{A}}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \quad (2\times 2)$$

$$\underline{\underline{B}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad (2\times 3) \qquad \underline{\underline{B}}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad (3\times 2)$$

II. Matrix Multiplication

$\underline{\underline{A}}$ $(m\times p)$ $\qquad$ $\underline{\underline{B}}$ $(p\times n)$

Then $\underline{\underline{A}}\,\underline{\underline{B}}$ is defined, $(m\times n)$

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \ (2\times 3) \qquad \underline{\underline{B}} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \ (3\times 2)$$

$$\underline{\underline{A}}\,\underline{\underline{B}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}_{2\times 2}$$

$(2\times 3)\ (3\times 2)$

(2×2)

$$= \begin{bmatrix} 1\cdot 1 + 2\cdot 2 + 3\cdot 1 & 1\cdot 1 + 2\cdot 1 + 3\cdot 1 \\ 4\cdot 1 + 5\cdot 2 + 6\cdot 1 & 4\cdot 1 + 5\cdot 1 + 6\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 6 \\ 20 & 15 \end{bmatrix} = \underline{\underline{A}}\,\underline{\underline{B}} \quad (2\times 2)$$

Ex: $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $\qquad$ $\underline{\underline{C}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$A \cdot C = $ ✗ NOT defined

$(2\times 3)(2\times 2)$ ✗

$C\,A = \begin{bmatrix} & & \end{bmatrix}$ (2×3)

$(2\times 2)(2\times 3)$ ✓

Ex: $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

which matrix multiplication is valid?

(a) AB $(3\times 3)(2\times 3)$ ✗

(b) BA $(2\times 3)(3\times 3)$ ✓

(c) both

(d) neither

III. Determinant:

If A is a 2×2 matrix then its determinant is

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Ex: Find $\det\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = 1\cdot 3 - 2\cdot 4 = 3 - 8 = -5$

For determinants of bigger square matrices, we can use Expansion by Minors

If A is an $n\times n$ matrix

Let A_{ij} to be the $(n-1)\times(n-1)$ matrix obtained by deleting its ith row and jth column

$$\sum^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

deleting its ...

$$\det(\underline{\underline{A}}) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(\underline{\underline{A}}_{ij})$$

Ex: $\underline{\underline{A}} = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 2 & 1 \\ -2 & 3 & 5 \end{bmatrix}$ let $i=1$

$$\det(\underline{\underline{A}}) = (-1)^2 \, 3 \cdot \det \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} + (-1)^3 \cdot (1) \cdot \det \begin{bmatrix} 4 & 1 \\ -2 & 5 \end{bmatrix} + (-1)^4 (-2) \det \begin{bmatrix} 4 & 2 \\ -2 & 3 \end{bmatrix}$$

$$(-1)^{1+1} a_{11} \det(\underline{\underline{A}}_{11}) + (-1)^{1+2} a_{12} \det(\underline{\underline{A}}_{12}) + (-1)^{1+3} a_{13} \det(\underline{\underline{A}}_{13})$$

$$= 3 \cdot \det \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} + (-1) \det \begin{bmatrix} 4 & 1 \\ -2 & 5 \end{bmatrix} + (-2) \det \begin{bmatrix} 4 & 2 \\ -2 & 3 \end{bmatrix}$$

$$= 3(2 \cdot 5 - 1 \cdot 3) + (-1)(4 \cdot 5 - 1(-2)) + (-2)(4 \cdot 3 - 2 \cdot (-2))$$

$$= 3(7) + (-1)(22) + (-2)(16)$$

$$= 21 - 22 - 32 = \boxed{-33 = \det(\underline{\underline{A}})}$$

IV. Matrix-Valued Functions:

Def: A matrix-valued function is a matrix in which each component is a function of t.

Ex: $\underline{x}(t) = \begin{bmatrix} t \\ 3t^2 \end{bmatrix}$ $\quad \underline{\underline{A}}(t) = \begin{bmatrix} \sin(t) & 7 \\ 0 & 8t^2+1 \end{bmatrix}$

Def: We say that a matrix fcn $\underline{\underline{A}}(t)$ is continuous at a point t if each of its elements is continuous

Def: The derivative of a matrix fcn is defined by elementwise differentiation

Def: The derivative of a ... elementwise differentiation

Ex: $\underline{x}(t) = \begin{bmatrix} t \\ t^2 \\ e^{-t} \end{bmatrix}$ then $\underline{x}'(t) = \dfrac{d\underline{x}}{dt} = \begin{bmatrix} 1 \\ 2t \\ -e^{-t} \end{bmatrix}$

$\underline{\underline{A}}(t) = \begin{bmatrix} \sin t & 1 \\ t & \cos t \end{bmatrix}$ $\qquad \underline{\underline{A}}'(t) = \dfrac{d\underline{\underline{A}}}{dt} = \begin{bmatrix} \cos t & 0 \\ 1 & -\sin t \end{bmatrix}$

sum and product rules:

$$\frac{d}{dt}(\underline{\underline{A}} + \underline{\underline{B}}) = \frac{d\underline{\underline{A}}}{dt} + \frac{d\underline{\underline{B}}}{dt}$$

$$\frac{d}{dt}(\underline{\underline{A}}\,\underline{\underline{B}}) = \underline{\underline{A}}\frac{d\underline{\underline{B}}}{dt} + \frac{d\underline{\underline{A}}}{dt}\underline{\underline{B}}$$

HW problem to show this

$$\frac{d}{dt}(c\,\underline{\underline{A}}) = c\,\frac{d\underline{\underline{A}}}{dt}$$

a scalar const.

V. First Order Linear Systems

System of ODEs → matrix form

$$\underset{(n\times 1)}{\frac{d\underline{x}}{dt}} = \underbrace{\underline{\underline{P}}(t)}_{\text{coefficient matrix}}\,\underline{x} + \underline{f}(t)$$

$(n\times 1) \qquad (n\times n)(n\times 1) \qquad (n\times 1)$

$(n\times 1)$

Ex: First order system:

$$x_1' = 4x_1 - 3x_2$$
$$x_2' = 6x_1 - 7x_2$$

matrix form:

x_2 ...

matrix form:

$$\frac{d\underline{x}}{dt} = \begin{bmatrix} 4 & -3 \\ 6 & 7 \end{bmatrix} \underline{x}$$

$\leftarrow$ here $\underline{f}(t) = \underline{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 4 & -3 \\ 6 & 7 \end{bmatrix}}_{\text{coefficient matrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Verify that $\underline{x}_1(t) = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$ and $\underline{x}_2(t) = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$

both solve the ODE

show: $\underline{x}_1' = \underline{P}(t)\underline{x}_1$

$$\underline{x}_1' = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix} \overset{?}{=} \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} 4\cdot 3e^{2t} - 3\cdot 2e^{2t} \\ 6\cdot 3e^{2t} - 7\cdot 2e^{2t} \end{bmatrix}$$

$$\overset{\checkmark}{=} \begin{bmatrix} (12-6)e^{2t} \\ (18-14)e^{2t} \end{bmatrix} = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix}$$

So yes, $\underline{x}_1$ solves the ODE.

Exercise $\longrightarrow$ show $\underline{x}_2$ solves the ODE

$$\underline{x}_2' = \underline{P}\,\underline{x}_2$$

VI. Associated Homogeneous Eqn

1st order linear system:

$$\frac{d\underline{x}}{dt} = \underline{\underline{P}}(t)\underline{x} + \underline{f}(t)$$

non homogeneous

Associated Homog. Eqn

$(\underline{f}(t) = 0)$

$$(*) \quad \frac{d\underline{x}}{dt} = \underline{\underline{P}}(t)\underline{x}$$

If $\underline{\underline{P}}(t)$ is an $n \times n$ matrix, then we expect (*) to have n linearly independent solutions:

$$\underline{x}_1(t), \ \underline{x}_2(t), \ \ldots, \ \underline{x}_n(t)$$

Since the system is linear, the Principle of Superposition applies

$$\underline{x}(t) = c_1 \underline{x}_1(t) + c_2 \underline{x}_2(t) + \ldots + c_n \underline{x}_n(t)$$

where $c_1, c_2, \ldots c_n$ are scalar constants

then $\underline{x}(t)$ also solves the ODE.

Proof:

$$\begin{aligned}
\underline{x}'(t) &= c_1 \underline{x}_1' + c_2 \underline{x}_2' + \ldots + c_n \underline{x}_n' \\
&= c_1 \underline{\underline{P}}\underline{x}_1 + c_2 \underline{\underline{P}} x_2 + \ldots + c_n \underline{\underline{P}} x_n \\
&= \underline{\underline{P}}\left(c_1 \underline{x}_1 + c_2 \underline{x}_2 + \ldots + c_n \underline{x}_n\right) \\
&= \underline{\underline{P}}\,\underline{x}
\end{aligned}$$

so $\underline{x}(t)$ solves the ODE.

Back to example:

$$\frac{d\underline{x}}{dt} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\underline{x}$$

has solns: $\underline{x}_1 = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$ and $\underline{x}_2 = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$

Then, the general soln is:

$$\underline{x}(t) = C_1 \underline{x}_1 + C_2 \underline{x}_2$$
$$= C_1 \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} + C_2 \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$

VII. Linear Independence:

We can use the Wronskian to determine linear indep.

If $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$ are solns to the ODE (*)
then the Wronskian

$$W(t) = \det \begin{bmatrix} | & | & & | \\ \underline{x}_1 & \underline{x}_2 & \cdots & \underline{x}_n \\ | & | & & | \end{bmatrix} \quad (n \times n)$$

If $W(t) \neq 0$ then $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$ are linearly independent.

Ex: $\underline{x}_1 = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$, $\underline{x}_2 = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$

$$W(t) = \det \begin{bmatrix} 3e^{2t} & e^{-5t} \\ 2e^{2t} & 3e^{-5t} \end{bmatrix} = (3e^{2t})(3e^{-5t}) - (e^{-5t})(2e^{2t})$$
$$= 9e^{-3t} - 2e^{-3t}$$
$$= 7e^{-3t} \neq 0$$

So $\underline{x}_1$ and $\underline{x}_2$ are linearly independent.