☆ The Eigenvalue Method for homogeneous systems

Warm up: Write the 1st order linear system

$$x_1' = 2x_1 - 3x_2$$
$$x_2' = -7x_1 + x_2$$

in matrix form.

$$\underline{x}' = \begin{bmatrix} 2 & -3 \\ -7 & 1 \end{bmatrix} \underline{x}$$

I. Eigenvalue Methods

$$\underline{x}' = \underline{\underline{A}}\,\underline{x} \qquad \underline{\underline{A}} \text{ is a } n\times n \text{ constant matrix}$$

Recall:

- 1st order linear ODE
  $x'(t) = \lambda x \longrightarrow x(t) = x_0 e^{\lambda t}$
- 2nd order linear ODE
  $ax'' + bx' + cx = 0$
  → assumed solns look like $x = e^{rt}$
  char eqn: $ar^2 + br + c = 0 \rightarrow$ roots r_1, r_2
  $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Try something similar for systems

Assume solns:

$$\underline{x}(t) = e^{\lambda t}\underline{v} = e^{\lambda t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \end{bmatrix}$$

Plug into ODE

$$\lambda \cancel{e^{\lambda t}}\,\underline{v} = \underline{x}' = \underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}(e^{\lambda t}\underline{v}) = \cancel{e^{\lambda t}}\,\underline{\underline{A}}\,\underline{v}$$

$$\lambda \underline{v} = \underline{\underline{A}}\,\underline{v}$$

Rewrite: $\boxed{(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{v} = 0}$ where $\underline{\underline{I}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Rewrite: $\boxed{(\underline{\underline{A}} - \underbrace{\lambda}_{\text{solve for } \lambda} \underline{\underline{I}})\underline{v} = 0}$ where $\underline{\underline{I}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ identity matrix

Note: This is the vector equivalent of the characteristic equation

λ is called an eigenvalue

$\underline{v}$ is called an eigenvector

Ex: $\underline{x}' = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \underline{x}$

solutions look like $\underline{x} = e^{\lambda t} \underline{v}$

Need to solve: $(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{v} = \underline{0}$

This system has a solution when

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0 \qquad \lambda \underline{\underline{I}} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\det \begin{bmatrix} 0-\lambda & 1 \\ 3 & 2-\lambda \end{bmatrix} = -\lambda(2-\lambda) - 1 \cdot 3$$

$$= \underbrace{\boxed{\lambda^2 - 2\lambda - 3 = 0}}_{\text{characteristic equation}}$$

$$(\lambda - 3)(\lambda + 1) = 0$$

$$\boxed{\begin{array}{c} \lambda = 3, -1 \\ \text{eigenvalues} \end{array}}$$

For each eigenvalue, there is corresponding eigen vector:

$\lambda_1 = 3 \longleftrightarrow \underline{v}^{(1)} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

To find $\underline{v}^{(1)}$ plug back into the char. eqn.

$$(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}})\underline{v}^{(1)} = \underline{0}$$

$$(\underline{A} - \lambda_1 \underline{\underline{I}})\underline{v}^{(1)} = \underline{0}$$

$$\begin{bmatrix} 0-3 & 1 \\ 3 & 2-3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3v_1 + v_2 = 0 \longrightarrow \boxed{v_2 = 3v_1}$$

$$3v_1 - v_2 = 0 \longrightarrow v_2 = 3v_1$$

NOTE: We expect there to be infinitely many solns b/c $\det\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} = 0$

so here, v_1 is a free variable

$$\underline{v}^{(1)} = \begin{bmatrix} v_1 \\ 3v_1 \end{bmatrix}$$ choose any value for v_1 (take $v_1 = 1$)

$$\boxed{\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \lambda_1 = 3}$$

Show that $\underline{x}^{(1)} = e^{\lambda_1 t}\underline{v}^{(1)} = e^{3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ solves the ODE

$$\underline{x}' = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}\underline{x}$$

$$3e^{3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}\left(e^{3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$$

$$e^{3t}\begin{bmatrix} 3 \\ 9 \end{bmatrix} \stackrel{\checkmark}{=} e^{3t}\begin{bmatrix} 3 \\ 9 \end{bmatrix}$$ so yes, $\underline{x}^{(1)}$ is a solution to the ODE

The second solution is:

$\underline{x}^{(2)} = e^{\lambda_2 t}\underline{v}^{(2)}$ $\quad \lambda_2 = -1$, Find $\underline{v}^{(2)} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

plug into the char. eqn

$$(\underline{A} - \lambda_2 \underline{\underline{I}})\underline{v}^{(2)} = \underline{0}$$

[illegible]

$(\underline{\underline{A}} - \lambda_2 \underline{\underline{I}}) =$

$$\begin{bmatrix} 0+1 & 1 \\ 3 & 2+1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 + v_2 = 0 \rightarrow v_2 = -v_1$$

$$3v_1 + 3v_2 = 0 \rightarrow v_2 = -v_1$$

v_1 is a free variable

choose $v_1 = 1$

$$\text{eigenvector } \underline{v}^{(2)} = \begin{bmatrix} v_1 \\ -v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\underline{x}^{(2)} = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Then, the general solution to the ODE:

$$\underline{x}(t) = C_1 e^{\lambda_1 t}\underline{v}^{(1)} + C_2 e^{\lambda_2 t}\underline{v}^{(2)}$$

$$= C_1 e^{3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + C_2 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

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eigenvalues

$\lambda_1 = 3$

$\lambda_2 = -1$

eigenvectors

$\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\underline{v}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\underline{Q}$: What does this mean geometrically?

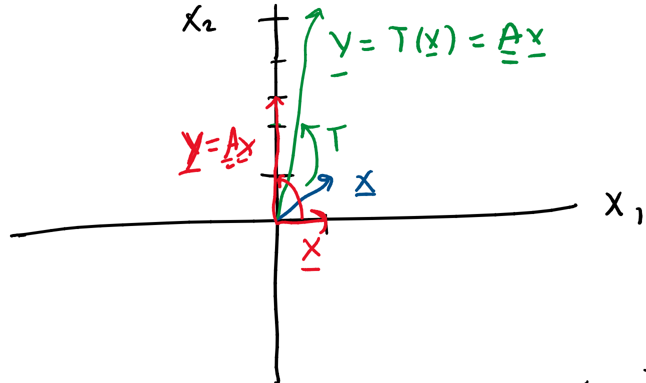
Think of $T(\underline{x}) = \underline{\underline{A}}\,\underline{x}$ is a transformation

$$\underline{x} \longrightarrow \boxed{T(\underline{x})} \longrightarrow \underline{y}$$

$$\begin{bmatrix} 1 \\ \cdots \end{bmatrix} \longrightarrow \boxed{T(\underline{x})} \longrightarrow \underline{y} = \underline{\underline{A}}\,\underline{x}$$

$$= \begin{bmatrix} 0 & 1 \\ \cdots \end{bmatrix}\begin{bmatrix} 1 \\ \cdots \end{bmatrix} = \begin{bmatrix} 1 \\ \cdots \end{bmatrix}$$

$$\begin{bmatrix}1\\1\end{bmatrix} \longrightarrow \boxed{T(\underline{x})} \longrightarrow \underline{y} = \underline{\underline{A}}\,\underline{x} = \begin{bmatrix}0 & 1\\3 & 2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\5\end{bmatrix}$$



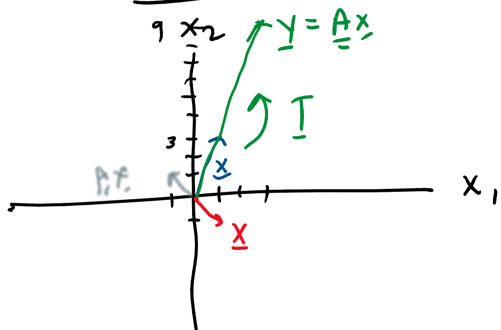
$\underline{y} = \underline{\underline{A}}\,\underline{x}$ stretches and rotates the vector $\underline{x}$

$$\begin{bmatrix}1\\0\end{bmatrix} \longrightarrow \boxed{T(\underline{x})} \longrightarrow \underline{y} = \begin{bmatrix}0 & 1\\3 & 2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix}$$

stretch + rotate $\underline{x}$

Q: What happens when $\underline{x} = \underline{v}^{(1)} = \begin{bmatrix}1\\3\end{bmatrix}$

$$\begin{bmatrix}1\\3\end{bmatrix} \longrightarrow \boxed{T(\underline{x})} \longrightarrow \underline{y} = \begin{bmatrix}0 & 1\\3 & 2\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}3\\9\end{bmatrix}$$



$\underline{\underline{A}}\,\underline{v}^{(1)} = 3\,\underline{v}^{(1)}$

stretch $\underline{v}^{(1)}$ by $\lambda_1 = 3$

(no rotation)

Similarly, when $\underline{x} = \underline{v}^{(2)} = \begin{bmatrix}1\\-1\end{bmatrix}$

$$\begin{bmatrix}1\\-1\end{bmatrix} \longrightarrow \boxed{T(\underline{x})} \longrightarrow \underline{y} = \begin{bmatrix}0 & 1\\3 & 2\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$$

flipped $\underline{v}^{(2)}$

$\underline{\underline{A}}\,\underline{v}^{(2)}$ multiplies $\underline{v}^{(2)}$ by $\lambda_2 = -1$

In terms of the ODE:

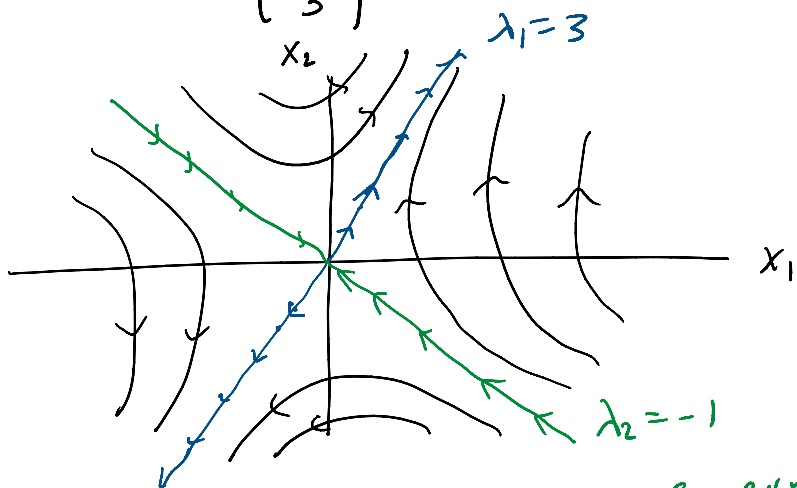
solution curves will grow exponentially

In terms of the ODE:

$$\underline{x}^{(1)}(t) = e^{3t}\begin{bmatrix}1\\3\end{bmatrix}$$

$$\frac{d\underline{x}^{(1)}}{dt}(t) = 3e^{3t}\begin{bmatrix}1\\3\end{bmatrix} = 3\,\underline{x}^{(1)}(t)$$

solution curves will grow exponentially along the vector $\begin{bmatrix}1\\3\end{bmatrix}$



$$\underline{x}^{(2)} = e^{-t}\begin{bmatrix}1\\-1\end{bmatrix}$$

solution curves exponentially decay along $\begin{bmatrix}1\\-1\end{bmatrix}$

Fill in remaining solution curves

General solution

$$\underline{x}(t) = c_1 e^{3t}\begin{bmatrix}1\\3\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}1\\-1\end{bmatrix}$$

Given initial conditions:

$$\underline{x}(0) = \begin{bmatrix}3\\1\end{bmatrix}$$

Find the particular solution

$$\underline{x}(0) = \begin{bmatrix}3\\1\end{bmatrix} = \left.\left(c_1 e^{3t}\begin{bmatrix}1\\3\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}1\\-1\end{bmatrix}\right)\right|_{t=0}$$

$$\begin{bmatrix}3\\1\end{bmatrix} = c_1\begin{bmatrix}1\\3\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix}$$

$$c_1 + c_2 = 3 \longrightarrow c_1 = 3 - c_2$$

$$c_1 + c_2 = 3 \longrightarrow c_1 = 3 - c_2$$
$$3c_1 - c_2 = 1$$
$$3(3-c_2) - c_2 = 1$$
$$9 - 3c_2 - c_2 = 1$$
$$-4c_2 = -8 \longrightarrow c_2 = 2, \; c_1 = 1$$

particular solution

$$\underline{x}_p(t) = e^{3t}\begin{bmatrix}1\\3\end{bmatrix} + 2e^{-t}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{bmatrix}e^{3t} + 2e^{-t}\\3e^{3t} - 2e^{-t}\end{bmatrix}$$

Eigenvalue Method:

Given: $\underline{x}' = \underline{\underline{A}}\,\underline{x}$ $\quad$ ($\underline{\underline{A}}$ is $n \times n$ matrix)

1. Plug in $\underline{x} = e^{\lambda t}\underline{v}$ into ODE to obtain
   $\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$ $\quad$ (Eigenvalue Problem)

2. Find n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$
   by solving $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$

3. For each λ_i find the corresponding eigenvector $\underline{v}^{(i)}$
   by solving: $(\underline{\underline{A}} - \lambda_i \underline{\underline{I}})\underline{v}^{(i)} = \underline{0}$

4. The general solution is:
   $\underline{x}(t) = c_1 e^{\lambda_1 t}\underline{v}^{(1)} + \cdots + c_n e^{\lambda_n t}\underline{v}^{(n)}$

5. Plug in initial conditions & solve $c_1, \ldots, c_n$
   $\underline{x}(0) = \underline{x}_0$