☆ Eigenvalue Method

Warm up: Find the eigenvalues of the matrix

$$\underline{\underline{A}} = \begin{bmatrix} 5 & 3 \\ 7 & 1 \end{bmatrix}$$

Ans: $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$ char. eqn $\longrightarrow$ solve for λ

$$\begin{vmatrix} 5-\lambda & 3 \\ 7 & 1-\lambda \end{vmatrix} = (5-\lambda)(1-\lambda) - 3\cdot 7 = 0$$

$$= \lambda^2 - \lambda - 5\lambda + 5 - 21$$

$$= \lambda^2 - 6\lambda - 16 = 0$$

$$(\lambda - 8)(\lambda + 2) = 0$$

$$\lambda = 8, -2$$

I. Eigenvalue Method:

$$\underline{x}'(t) = \underline{\underline{A}}\,\underline{x}$$

1. Find the eigenvalues λ

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$$

2. Find the eigenvectors $\underline{v}$

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{v} = \underline{0}$$

3. General solution:

$$\underline{x}(t) = c_1 e^{\lambda_1 t}\underline{v}^{(1)} + c_2 e^{\lambda_2 t}\underline{v}^{(2)}$$

Q: What happens if the eigenvalues are complex-valued.

II. Complex Eigenvalues

Ex: $\underline{x}' = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{x}$

Ex: $\underline{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \underline{x}$

eigenvalues: $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = 0$$

$$(1-\lambda)^2 = -1$$

$$1-\lambda = \pm i$$

$$\lambda = 1 \pm i$$

NOTE: complex eigenvalues always appear in conjugate pairs

eigenvectors: $(\underline{A} - \lambda \underline{\underline{I}})\underline{v} = \underline{0}$

$\lambda_1 = 1+i$ $\quad \begin{bmatrix} 1-(1+i) & 1 \\ -1 & 1-(1+i) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-iv_1 + v_2 = 0 \longrightarrow v_2 = iv_1$$

$$-v_1 - iv_2 = 0 \longrightarrow v_2 = \frac{-v_1}{i} = iv_1$$

so v_1 is a free variable, choose $v_1 = 1$

$$\underline{v}^{(1)} = \begin{bmatrix} v_1 \\ iv_1 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$\lambda_2 = 1-i$ $\quad \rightarrow \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\rightarrow iv_1 + v_2 = 0 \longrightarrow v_2 = -iv_1$$

so v_1 is a free variable, choose $v_1 = 1$

$$v^{(2)} = \begin{bmatrix} v_1 \\ \end{bmatrix} = \begin{bmatrix} 1 \\ \end{bmatrix}$$

So v_1 ...

$$\underline{v}^{(2)} = \begin{bmatrix} v_1 \\ -iv_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

General solution:

$$\underline{x}(t) = C_1 e^{(1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} + C_2 e^{(1-i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

<u>Note</u>: eigenvectors are <u>also</u> conjugate pairs

<u>Rewrite</u>:

Euler's formula

$$e^{it} = \cos t + i \sin t$$

$$e^{(1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} = e^t e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= e^t (\cos t + i \sin t) \begin{bmatrix} 1 \\ i \end{bmatrix} = e^t \begin{bmatrix} \cos t + i \sin t \\ i \cos t - \sin t \end{bmatrix}$$

$$= e^t \left\{ \underbrace{\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}_{\underline{u}} + i \underbrace{\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}_{\underline{v}} \right\} = \boxed{e^t \{ \underline{u} + i \underline{v} \}}$$

Similarly, we find

$$e^{(1-i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \boxed{e^t \{ \underline{u} - i \underline{v} \}}$$

fundamental solutions are also complex conjugates

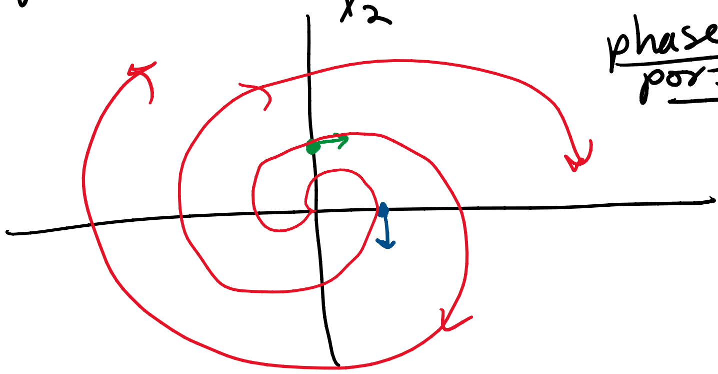
So to get real-valued solutions:

$$\underline{x}(t) = c_1 e^t \underline{u} + c_2 e^t \underline{v}$$

$$\boxed{\underline{x}(t) = c_1 e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}$$

... solution

For a 2D system, we can represent the solution graphically in the <u>phase</u> <u>plane</u>:



<u>phase portrait</u>

$$\underline{x}_1 = \underbrace{e^{t}}_{\text{amplitude grows exponentially as } t\to\infty} \underbrace{\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}_{\text{oscillate between } -1 \text{ and } +1}$$

$\rightarrow$ solution curves spiral outwards

Q: Clockwise or counter clockwise?

$\underline{u} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$ $\quad \underline{u}' = \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix}$ $\quad u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad u'(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\underline{v} = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$ $\quad \underline{v}' = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$ $\quad \underline{v}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \underline{v}'(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

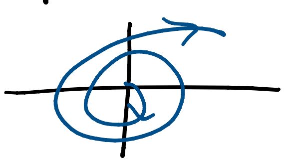
move in clockwise direction

phase portrait is an outward clockwise spiral.

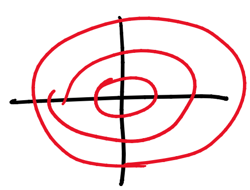
<u>complex eigenvalues</u>:

phase portrait is a spiral

$\mathrm{Re}(\lambda) > 0$ spiral out

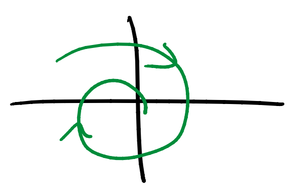


$\mathrm{Re}(\lambda) = 0$



(center)

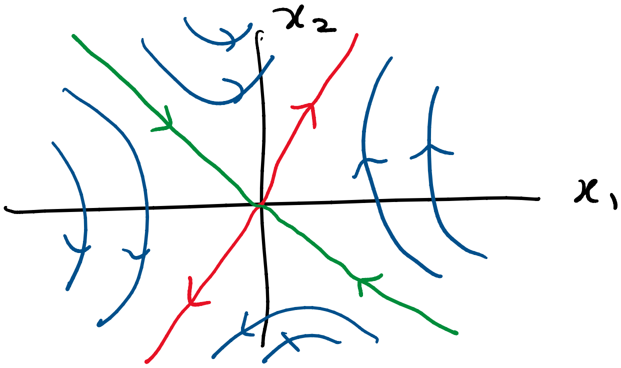
$\mathrm{Re}(\lambda) < 0$ spiral in



Ex (2): Last lecture

$$\underline{x}' = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \underline{x}$$

$$\left[\begin{array}{l} \lambda_1 = 3 \\ \underline{v}^{(1)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{array}\right. \qquad \left[\begin{array}{l} \lambda_2 = -1 \\ \underline{v}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{array}\right.$$



phase portrait is called a <u>saddle point</u>

III. 3x3 system:

Ex:
$$\underline{x}' = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} \underline{x}$$

3x 3 system
3 eigenvalues
3 eigenvector

eigenvalues: $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 7-\lambda & 1 \\ 2 & 1 & 7-\lambda \end{vmatrix} = 0$$

expansion by minors

$$= (1-\lambda)\begin{vmatrix} 7-\lambda & 1 \\ 1 & 7-\lambda \end{vmatrix} - (2)\begin{vmatrix} 2 & 1 \\ 2 & 7-\lambda \end{vmatrix} + (2)\begin{vmatrix} 2 & 7-\lambda \\ 2 & 1 \end{vmatrix}$$

$$= (1-\lambda)\left[(7-\lambda)^2 - 1\right] - 2\left[2(7-\lambda) - 2\right] + 2\left[2 - 2(7-\lambda)\right]$$

$$= (1-\lambda)(7-\lambda)^2 - (1-\lambda) - 4(7-\lambda) + 4 + 4 - 4(7-\lambda)$$

$$= (1-\lambda)(49 - 14\lambda + \lambda^2) - (1-\lambda) - 8(7-\lambda) + 8$$

$$= 49 - 14\lambda + \lambda^2 - 49\lambda + 14\lambda^2 - \lambda^3 - 1 + \lambda - 56 + 8\lambda + 8$$

collect like terms

$= 49 - 14\lambda + \lambda^{2} - 49\lambda \ldots$

collect like terms

$$= -\lambda^3 + (1+14)\lambda^2 + (-14-49+1+8)\lambda + (49-1-56+8)$$

$$= -\lambda^3 + 15\lambda^2 - 54\lambda + 0$$

$$-\lambda(\lambda^2 - 15\lambda + 54) = 0$$

$$-\lambda(\lambda-9)(\lambda-6) = 0$$

$$\boxed{\lambda = 0, 6, 9}$$

eigenvectors:

$\boxed{\lambda_1 = 0}$ $\qquad (\underline{\underline{A}} - 0\underline{\underline{I}})\underline{v} = \underline{0}$

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow v_1 + 2v_2 + 2v_3 = 0 \longrightarrow v_1 = -2v_2 - 2v_3$

$\rightarrow 2(-2v_2 - 2v_3) + 7v_2 + v_3 = 0$

$$(-4+7)v_2 + (-4+1)v_3 = 0$$

$$3v_2 - 3v_3 = 0 \longrightarrow v_2 = v_3$$

$$v_1 = -2v_3 - 2v_3 = -4v_3$$

so v_3 is a free variable, choose $v_3 = 1$

$$\underline{v}^{(1)} = \begin{bmatrix} -4v_3 \\ v_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$$

fundamental soln: $\underline{x}^{(1)} = e^{0\cdot t}\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$

$\boxed{\lambda_2 = 6}$ $\quad (\underline{\underline{A}} - 6\underline{\underline{I}})\underline{v} = \underline{0}$

$v_1 = 0$
$v_2 = -v_3$

$$\begin{bmatrix} -5 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2v_1 + v_2 + v_3 = 0 \longrightarrow v_3 = -2v_1 - v_2$$

$$-5v_1 + 2v_2 + 2(-2v_1 - v_2) = 0$$

$$(-5-4)v_1 + \cancel{(2-2)}^{\,0} v_2 = 0$$

$$-9v_1 = 0 \longrightarrow v_1 = 0$$

$$v_3 = -v_2$$

so v_2 is a free variable, choose $v_2 = 1$

$$\underline{v}^{(2)} = \begin{bmatrix} 0 \\ v_2 \\ -v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

fundamental soln: $\quad \underline{x}^{(2)} = e^{6t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$\boxed{\lambda_3 = 9}$ $\quad (\underline{\underline{A}} - 9\underline{\underline{I}})\underline{v} = \underline{0}$

$$\begin{bmatrix} -8 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2v_1 - 2v_2 + v_3 = 0 \longrightarrow v_3 = -2v_1 + 2v_2$$

$$2v_1 + v_2 - 2(-2v_1 + 2v_2) = 0$$

$$(2+4)v_1 + (1-4)v_2 = 0$$

$$6v_1 - 3v_2 = 0$$

$$v_2 = 2v_1$$

$$v_3 = -2v_1 + 4v_1 = 2v_1$$

$$v_2 = 2v_1$$
$$v_3 = -2v_1 + 4v_1 = 2v_1$$

So v_1 is a free variable, choose $v_1 = 1$

$$\underline{v}^{(3)} = \begin{bmatrix} v_1 \\ 2v_1 \\ 2v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

fundamental solution:

$$\underline{x}^{(3)} = e^{9t} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

General solution:

$$\underline{x}(t) = c_1 \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{9t} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\underline{x}(t) = \begin{bmatrix} -4c_1 + c_3 e^{9t} \\ c_1 + c_2 e^{6t} + 2c_3 e^{9t} \\ c_1 + c_2 e^{6t} + 2c_3 e^{9t} \end{bmatrix}$$