☆ A Gallery of Solution Curves

Warm up: Find the trace T of the matrix $\underline{\underline{A}}$

$$\underline{\underline{A}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Ans: $T = \text{trace}(\underline{\underline{A}}) = a + d$

For a 2D linear systems:

$$\underline{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \underline{x}$$

the phase portrait depends on the eigenvalues and eigenvectors of $\underline{\underline{A}}$

eigenvectors: $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc$$
$$= \lambda^2 - a\lambda - d\lambda + ad - bc = 0$$

$$\lambda^2 - \underbrace{(a+d)}_{T = \text{trace}(\underline{\underline{A}})}\lambda + \underbrace{(ad-bc)}_{D = \det(\underline{\underline{A}})} = 0$$

$$\lambda^2 - T\lambda + D = 0$$

eigenvalues are roots of 2nd degree polynomial.

Three major cases:

I. Real distinct λ

II. Repeated λ with $k=2$

III. Complex λ

can broken into subcases.

III. Complex λ

Each case can broken into subcases.

I. Real distinct λ:

(a) λ opposite signs:

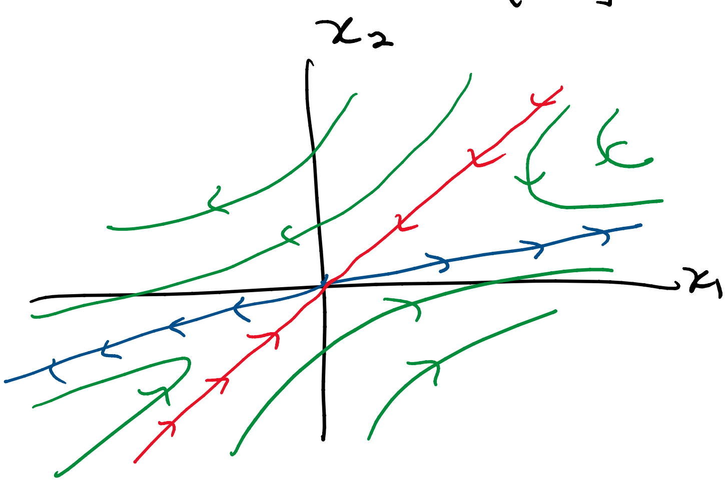
Ex: $\underline{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -2 \end{bmatrix} \underline{x}$

$\lambda_1 = 2 \quad \underline{v}^{(1)} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$\lambda_2 = -1 \quad \underline{v}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

general solution:

$\underline{x}(t) = c_1 e^{2t} \begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



1. Draw the eigenvectors
2. Draw connecting curves

$\boxed{\text{saddle point}}$

1 ev points in
1 ev points out

(b) λ both negative

Ex: $\underline{x}' = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix} \underline{x}$

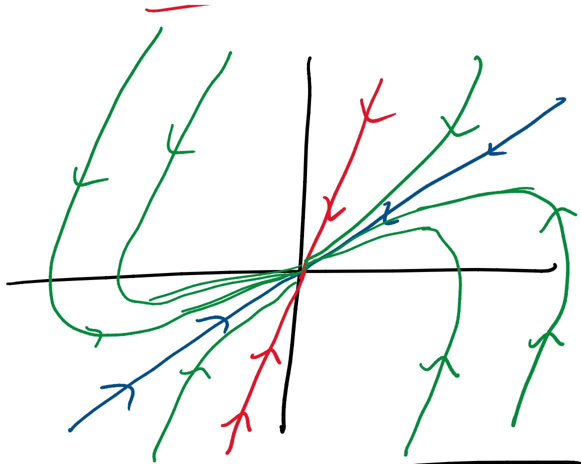
$\lambda_1 = -1 \quad \underline{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -2 \quad \underline{v}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

general solution:

$\underline{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

1. Draw eigenvectors



1. Draw eigenvectors
2. Limit as $t \to \infty$
   e^{-t} "dominate"
   $\underline{x} \to$ parallel $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
3. Limit as $t \to -\infty$
   e^{-2t} dominates
   $\underline{x} \to$ parallel $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\boxed{\text{improper nodal sink}}$

- multiple solns approach origin
- tangent to the same line $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c) λ both positive

$\boxed{\text{improper nodal source}}$

Same as (b) but with arrows flipped

as $t \to \infty$, $\underline{x} \to +\infty$

(d) one zero eigenvalue

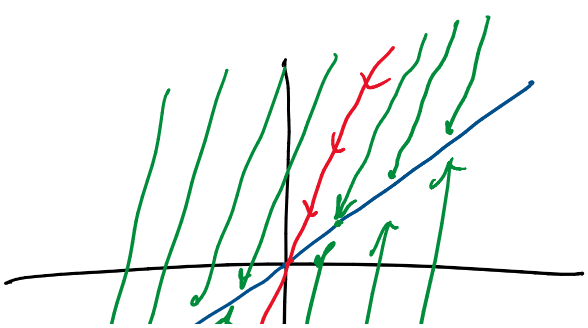
Ex: $\underline{x}' = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \underline{x}$

$\lambda_1 = 0 \qquad \underline{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

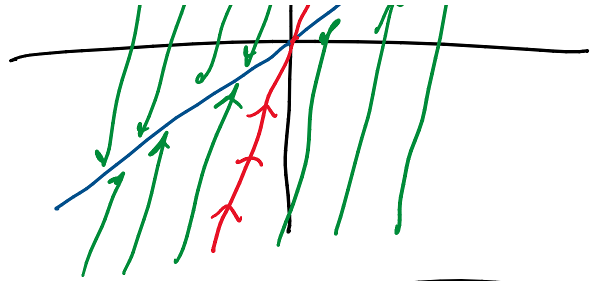
$\lambda_2 = -1 \qquad \underline{v}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

general solution:

$$\underline{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



1. Draw eigenvectors
2. Limits at $t \to \infty$
   $e^{-t} \to 0$
   $\underline{x} \to c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



$\underline{x} \to c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3. limit as $t \to -\infty$

$\underline{x} \to$ parallel $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\boxed{\text{parallel lines}}$

$\lambda_2 < 0 \to$ stop on $\underline{v}^{(1)}$

$\lambda_2 > 0 \to$ emanate from $\underline{v}^{(1)}$

II. Real repeated λ:

(a) If λ is complete

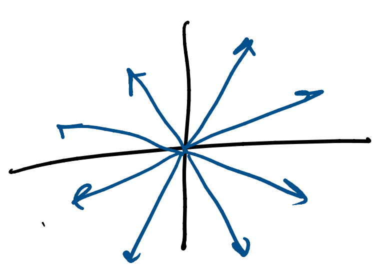
geometric mult p = algebraic mult k

# of eigenvectors = 2

2 linearly indep eigenvectors $\underline{v}^{(1)}$ and $\underline{v}^{(2)}$

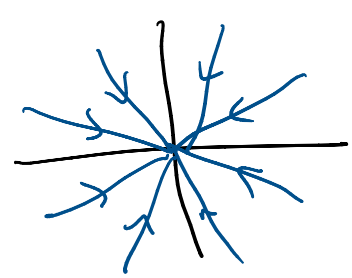
General solution: $\underline{x}(t) = c_1 e^{\lambda t} \underline{v}^{(1)} + c_2 e^{\lambda t} \underline{v}^{(2)}$

if $\lambda > 0$



proper nodal source

if $\lambda < 0$



proper nodal sink

(b) If λ is defective

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geometric mult p < algebraic mult $k = 2$

# of eigenvectors = 1

Ex: $\underline{x}' = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \underline{x}$ $\quad \lambda = 3$ w/ alg mult $k=2$

$\lambda = 3$ $\quad \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad v_1 + v_2 = 0$, $v_2 = -v_1$ $\quad \underline{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

one eigenvector $\Rightarrow$ $p = 1$

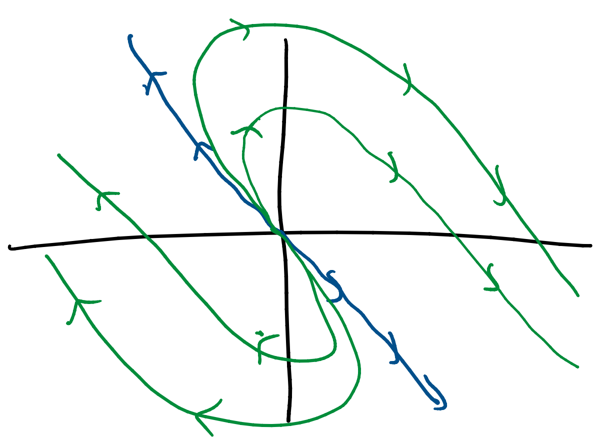
Need to find generalized eigenvector $\underline{u}$

$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \longrightarrow u_1 + u_2 = 1$, $u_2 = 1 - u_1$ $\quad \underline{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

General solution:

$$\underline{x}(t) = c_1 e^{\lambda t} \underline{v} + c_2 e^{\lambda t} \{ t\underline{v} + \underline{u} \}$$

$$\underline{x}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \left\{ t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$



1. Draw eigenvector

2. limit as $t \to \infty$
te^{3t} dominates
$\underline{x} \to$ parallel $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

improper nodal source

(if $\lambda < 0$, sink)

(c) $\lambda = 0$ repeated

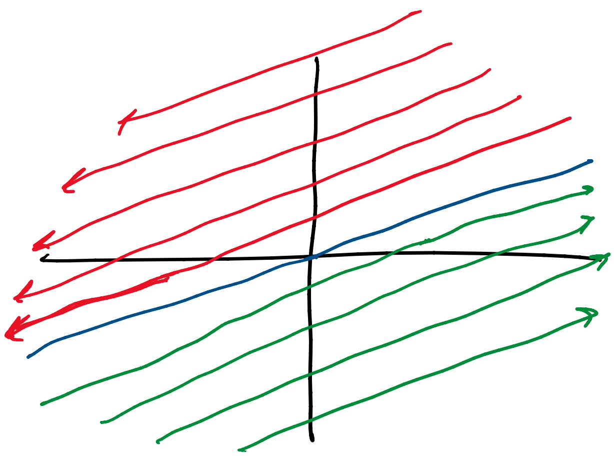
Ex: $\underline{x}' = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \underline{x}$

char eqn: $\lambda^2 = 0$

$\lambda = 0$ w/ $k=2$

$\lambda = 0$ $\underline{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $p=1$, generalized eigenvector $\underline{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

general solution: $\underline{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \left\{ t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

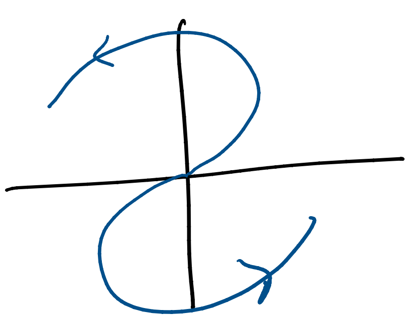


1. Draw eigenvector
2. limit as $t \to \infty$
   t dominates
   $\underline{x} \to$ parallel $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
3. plug in $c_1 = 0$, $c_2 = 1$
   $\underline{x} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}$
4. plug in $c_1 = 0$, $c_2 = -1$
   $\underline{x} = -t \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

variation on **parallel lines**

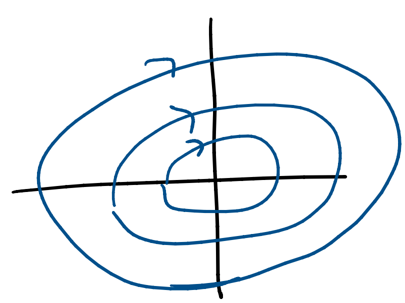
III. Complex λ: $\lambda = a \pm bi$

(a) $Re(\lambda) > 0$



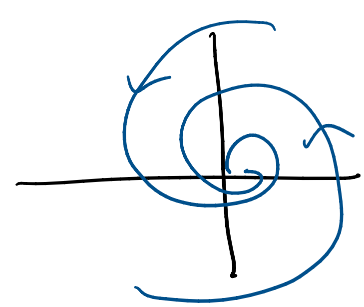
spiral source

(b) $Re(\lambda) = 0$



center

(c) $Re(\lambda) < 0$



spiral sink

$\rightarrow$ look at solution to determine direction of rotation.

$$\lambda = a \pm bi \qquad \underline{x}(t) = e^{at}\left(c_1 \underline{u} + c_2 \underline{v}\right)$$

evaluate: $\left\{ \begin{array}{ll} \underline{u}(0) & \underline{v}(0) \\ \underline{u}'(0) & \underline{v}'(0) \end{array} \right.$

Ex: $\underline{x}' = \begin{bmatrix} 6 & -17 \\ 8 & -6 \end{bmatrix} \underline{x} \qquad \lambda = \pm 10i$

$\lambda = 10i$
$$\begin{bmatrix} 6-10i & -17 \\ 8 & -6-10i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$8v_1 - (6+10i)v_2 = 0$$

$$v_1 = \frac{(6+10i)}{8} v_2 = \frac{(3+5i)}{4} v_2$$

v_2 is a free variable, choose $v_2 = 4$

$$\underline{v}^{(1)} = \begin{bmatrix} 3+5i \\ 4 \end{bmatrix}$$

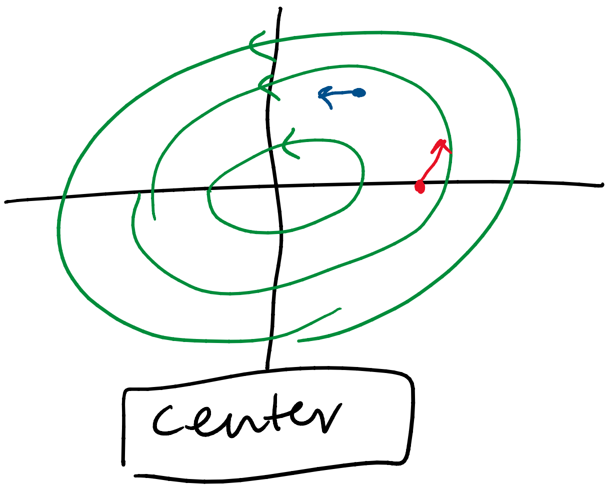
$$\underline{x}^{(1)} = e^{10it} \begin{bmatrix} 3+5i \\ 4 \end{bmatrix} = \left(\cos(10t) + i\sin(10t)\right) \begin{bmatrix} 3+5i \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cos(10t) + 3i\sin(10t) + 5i\cos(10t) - 5\sin(10t) \\ 4\cos(10t) + 4i\sin(10t) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 3\cos(10t) - 5\sin(10t) \\ 4\cos(10t) \end{bmatrix}}_{\underline{u}} + i \underbrace{\begin{bmatrix} 3\sin(10t) + 5\cos(10t) \\ 4\sin(10t) \end{bmatrix}}_{\underline{v}}$$

general solution: $\underline{x}(t) = c_1 \underline{u} + c_2 \underline{v}$

$$\underline{x}(t) = c_1 \begin{bmatrix} 3\cos(10t) - 5\sin(10t) \\ 4\cos(10t) \end{bmatrix} + c_2 \begin{bmatrix} 3\sin(10t) + 5\cos(10t) \\ 4\sin(10t) \end{bmatrix}$$



1. center b/c $Re(\lambda) = 0$

2. $\underline{u}(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $\underline{u}'(0) = \begin{bmatrix} -50 \\ 0 \end{bmatrix}$

3. $\underline{v}(0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ $\underline{v}'(0) = \begin{bmatrix} 30 \\ 40 \end{bmatrix}$

ccw direction