☆ Multiple Eigenvalues:

Warm up: Find the eigenvalues of the matrix

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 0 \\ 7 & 2 \end{bmatrix}$$

Ans: $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$

$$\begin{vmatrix} 2-\lambda & 0 \\ 7 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 0\cdot 7 = 0$$

$$\lambda = 2$$

multiplicity $k = 2$

I. Multiple Eigenvalue Solutions:

In Chap 3, we solved,

$$\boxed{y'' - 6y' + 9y = 0}$$

We can convert to a system of ODE

$x_1 = y \qquad x_2 = y'$

$$\begin{cases} x_1' = x_2 \\ x_2' = 6x_2 - 9x_1 \end{cases}$$

matrix form: $\boxed{\underline{x}' = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix} \underline{x}}$

these 2 ODEs are equivalent

	1D	2D
eqn:	$y'' - 6y' + 9y = 0$	$\underline{x}' = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix} \underline{x}$
char eqn	$r^2 - 6r + 9 = 0$	$\begin{vmatrix} -\lambda & 1 \\ -9 & 6-\lambda \end{vmatrix} = 0$ $\lambda^2 - 6\lambda + 9 = 0$
rts/	$(r-3)^2 = 0$	$(\lambda-3)^2 = 0$

roots/ eigenvalues	$(r-3)^2=0$ $r=3$ multiplicity $k=2$ (repeated root)	$(\lambda-3)^2=0$ $\lambda=3$ algebraic multiplicity $k=2$ (repeated eigenvalue)
fundamental solns:	$y_1=e^{3t}$ Need a 2nd lin. indep soln $\rightarrow$ multiply by t $y_2=te^{3t}$	$\lambda=3$ $(\underline{\underline{A}}-3\underline{\underline{I}})\underline{v}=\underline{0}$ $\begin{bmatrix}-3 & 1\\ -9 & 3\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ $-3v_1+v_2=0 \rightarrow v_2=3v_1$ $\underline{v}^{(1)}=\begin{bmatrix}1\\ 3\end{bmatrix}$ * only 1 eigenvector geometric multiplicity $=1$ Need to find a 2nd lin. indep vector $\underline{u}$

NOTE: when:

geometric multiplicity (# of eigenvectors corr. λ) $<$ algebraic multiplicity (# times λ is repeated in char eqn)

We say that the eigenvalue λ is defective

2D example:

one fundamental soln: $\underline{x}^{(1)}=e^{3t}\begin{bmatrix}1\\ 3\end{bmatrix}$

Need to find a generalized eigenvector $\underline{u}$

Need to find a <u>generalized</u> <u>eigenvector</u> $\underline{u}$

Solve: $(\underline{A} - \lambda \underline{\underline{I}})\underline{u} = \underline{v}$

<u>NOTE</u>: this means also solves:

$$(\underline{A} - \lambda \underline{\underline{I}})^2 \underline{u} = \underline{0}$$

$$(\underline{A} - \lambda \underline{\underline{I}})^2 \underline{u} = (\underline{A} - \lambda \underline{\underline{I}}) \underbrace{(\underline{A} - \lambda \underline{\underline{I}}) \underline{u}}_{= \underline{v}} = (\underline{A} - \lambda \underline{\underline{I}}) \underline{v} = \underline{0}$$

want to solve:

$$(\underline{A} - 3\underline{\underline{I}})\underline{u} = \underline{v}$$

$$\begin{bmatrix} -3 & 1 \\ -9 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$-3u_1 + u_2 = 1 \longrightarrow u_2 = 1 + 3u_1$$

here, u_1 is a free variables choose $u_1 = 1$

$$\underline{u} = \begin{bmatrix} u_1 \\ 1 + 3u_1 \end{bmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$ generalized eigenvector.

Second fundamental solution

$$\underline{x}^{(2)} = e^{3t}(\underline{v}t + \underline{u})$$

$$= e^{3t}\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} t + \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\} = e^{3t} \begin{bmatrix} t+1 \\ 3t+4 \end{bmatrix}$$

General solution:

$$\boxed{\underline{x}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} t+1 \\ 3t+4 \end{bmatrix}}$$

... case:

compare this to 1D case:

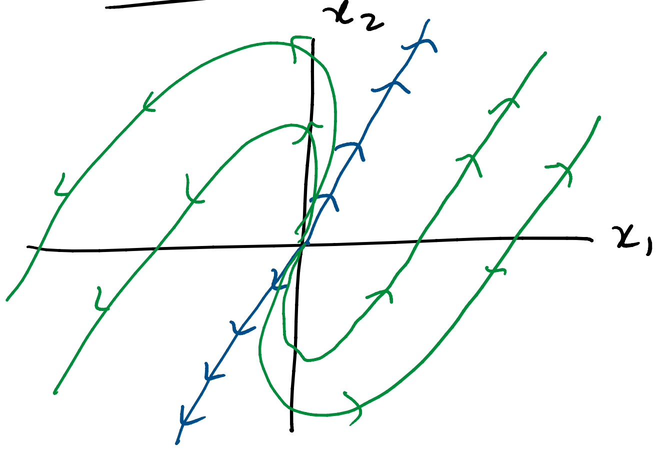
$$y(t) = C_1 e^{3t} + C_2 t e^{3t}$$

$$x_1 = y = B_1 e^{3t} + B_2 (t+1) e^{3t}$$

$$= (B_1 + B_2) e^{3t} + B_2 t e^{3t}$$

equivalent solutions

phase portrait:



improper node

1. Draw eigenvector $\underline{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

2. $\lambda = 3 > 0$ arrows point out

3. As $t \to \infty$ $e^{3t} t \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ dominates goes parallel to $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

II. 3x3 Example:

$$\underline{x}' = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \underline{x}$$

eigenvalues: $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$

$$\begin{vmatrix} 1-\lambda & 3 & -1 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} - (3) \begin{vmatrix} 0 & 0 \\ 0 & 2-\lambda \end{vmatrix}$$

$$+ (-1) \begin{vmatrix} 0 & 2-\lambda \\ 0 & 0 \end{vmatrix}$$

$$= (1-\lambda)\left[(2-\lambda)^2 - 0\right] - 3\left[\cancel{0-0}\right]^{0} + (-1)\left[\cancel{0-0}\right]^{0}$$

$$= (1-\lambda)\left[(2-\lambda)^2 - 0\right] - 3\left[\cancelto{0}{0-0}\right] + (-1)\left[\cancelto{0}{0-0}\right]$$

$$= (1-\lambda)(2-\lambda)^2 = 0$$

$$\boxed{\begin{array}{ll} \lambda = 1 & \text{alg. mult } k = 1 \\ \lambda = 2 & \text{alg. mult } k = 2 \end{array}}$$

eigenvectors:

$\boxed{\lambda_1 = 1}$ $\quad (\underline{\underline{A}} - \underline{\underline{I}})\underline{v} = \underline{0}$

$$\begin{bmatrix} 0 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{l} \rightarrow 3v_2 - v_3 = 0 \\ \rightarrow v_2 = 0 \\ \rightarrow v_3 = 0 \end{array}$$

v_1 is a free variable, choose $v_1 = 1$

$$\underline{v}^{(1)} = \begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{\lambda_2 = 2}$ $\quad (\underline{\underline{A}} - 2\underline{\underline{I}})\underline{v} = \underline{0}$

$$\begin{bmatrix} -1 & 3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{l} \\ \leftarrow \\ \leftarrow \end{array}$$

nothing

1 eqn, 3 unknowns $\downarrow$ 2 free variables

$$-v_1 + 3v_2 - v_3 = 0$$

$$v_1 = 3v_2 - v_3$$

v_2 and v_3 are free variables

$$\underline{v} = \begin{bmatrix} 3v_2 - v_3 \\ v_2 \\ v_3 \end{bmatrix}$$

choose $v_2 = 1$, $v_3 = 0$

$$\underline{v}^{(2)} = \begin{bmatrix} 3 \\ \end{bmatrix}$$

choose $v_2 = 0$, $v_3 = 1$

$$\underline{v}^{(3)} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\underline{v}^{(2)} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \qquad \longleftrightarrow \qquad \underline{v}^{(3)} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

these are linearly indep.

2 linearly indep eigenvectors for $\lambda_2 = 2$

geometric multiplicity $= 2 =$ algebraic multiplicity

Then the eigenvalue $\lambda = 2$ is called <u>complete</u>

Do <u>NOT</u> need to find a generalized eigenvector $\underline{u}$

General solution:

$$\boxed{\underline{x}(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

III <u>Summary</u>: Repeated eigenvalues

k = algebraic mult. = # of times λ is repeated

p = geometric mult = # of lin. indep eigenvectors

case 1	case 2
$p = k$	$p < k$ $\quad (k=2)$
λ is complete	λ is defective

λ is complete

eigenvectors
$\underline{v}_1, \ldots, \underline{v}_p$

gen soln:
$\underline{x}(t) = c_1 e^{\lambda t} \underline{v}_1 + \ldots + c_p e^{\lambda t} \underline{v}_p$

λ is defective

eigen vector $\underline{v}$
generalized eigenvector $\underline{u}$

$\underline{x}(t) = c_1 e^{\lambda t} \underline{v} + c_2 e^{\lambda t}(t\underline{v} + \underline{u})$