☆ Matrix Exponentials

Warm up: Write down the Taylor series expansion of the function e^t (around $t=0$)

Ans: $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$

and $e^{kt} = 1 + kt + \frac{(kt)^2}{2!} + \frac{(kt)^3}{3!} + \dots$

I. Exponential Matrix:

Linear systems of ODE:

$$\underline{x}' = \underline{\underline{A}}\,\underline{x}$$

General Principle: Linear ODE $\longrightarrow$ solns are exponentials

Eigenvalue method: guessed $\underline{x} = e^{\lambda t}\underline{v}$

Today: given the IVP:

$$\underline{x}' = \underline{\underline{A}}\,\underline{x} \qquad \underline{x}(0) = \underline{x}_0$$

We can write the solution as:

$$\underline{x}(t) = \underbrace{e^{\underline{\underline{A}}t}}_{\text{Exponential of a matrix}}\,\underline{x}_0$$

1D:

$y' = ay \quad y(0) = y_0$

$y(t) = y_0 e^{at}$

Def: If $\underline{\underline{A}}$ is an $n \times n$ matrix, then the exponential matrix is defined by the series:

$$e^{\underline{\underline{A}}t} = \underline{\underline{I}} + t\underline{\underline{A}} + \frac{t^2}{2!}\underline{\underline{A}}^2 + \frac{t^3}{3!}\underline{\underline{A}}^3 + \dots + \frac{t^n}{n!}\underline{\underline{A}}^n + \dots$$

where $\underline{\underline{I}}$ is the identity matrix

Ex: let $A = \begin{bmatrix} a & 0 \\ \cdot & \cdot \end{bmatrix}$ Find $e^{\underline{\underline{A}}t}$

Ex: Let $\underline{\underline{A}} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ Find $e^{\underline{\underline{A}}t}$

$\underline{\underline{I}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \underline{\underline{A}}^2 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$

$\underline{\underline{A}}^3 = \underline{\underline{A}}\,\underline{\underline{A}}^2 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} = \begin{bmatrix} a^3 & 0 \\ 0 & b^3 \end{bmatrix}$ see a pattern

$$\underline{\underline{A}}^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$$

so $e^{\underline{\underline{A}}t} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} + \frac{t^3}{3!}\begin{bmatrix} a^3 & 0 \\ 0 & b^3 \end{bmatrix} + \dots + \frac{t^n}{n!}\begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix} + \dots$

$$= \begin{bmatrix} 1 + at + \frac{1}{2!}a^2t^2 + \frac{t^3}{3!}a^3 + \dots & 0 \\ 0 & 1 + bt + \frac{t^2}{2!}b^2 + \frac{t^3}{3!}b^3 + \dots \end{bmatrix}$$

$$\boxed{e^{\underline{\underline{A}}t} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix}}$$

NOTE: when $\underline{\underline{A}}$ is a diagonal, then $e^{\underline{\underline{A}}}$ is just exponentiates each element on the diagonal

$$\underline{\underline{D}} = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix} \qquad e^{\underline{\underline{D}}t} = \begin{bmatrix} e^{a_1 t} & & & 0 \\ & e^{a_2 t} & & \\ & & \ddots & \\ 0 & & & e^{a_n t} \end{bmatrix}$$

Ex: $\underline{\underline{A}} = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$ Find $e^{\underline{\underline{A}}t}$

$$\underline{\underline{A}}^2 = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{A}^3 = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underline{0}$$

so $\underline{A}^4 = \underline{0}$, $\underline{A}^5 = \underline{0}$, so $\underline{A}^n = \underline{0}$ for all $n \geq 3$

<u>Def</u>: A matrix is called <u>nipotent</u> if for some power n, $\underline{A}^n = \underline{0}$

Exponential

$$e^{\underline{A}t} = \underline{I} + \underline{A}t + \frac{1}{2!}\underline{A}^2t^2 + \frac{1}{3!}\underline{A}^3t^3 + \cdots$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t\begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix} 0 & 0 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{e^{\underline{A}t} = \begin{bmatrix} 1 & 3t & 4t + 9t^2 \\ 0 & 1 & 6t \\ 0 & 0 & 1 \end{bmatrix}}$$

<u>Ex</u>: $\underline{A} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ Find $e^{\underline{A}t}$

<u>NOTE</u>: $\underline{A} = \underline{D} + \underline{B}$

$$= \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{\text{diagonal}} + \underbrace{\begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{nilpotent}}$$

$$e^{\underline{A}t} = e^{(\underline{D}+\underline{B})t} = e^{\underline{D}t}\, e^{\underline{B}t}$$

$$e^{\underline{\underline{A}}t} = e^{\cdots} = e^{\cdots} e^{\cdots}$$

$$= \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 3t & 4t+9t^2 \\ 0 & 1 & 6t \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{e^{\underline{\underline{A}}t} = \begin{bmatrix} e^{2t} & 3te^{2t} & (4t+9t^2)e^{2t} \\ 0 & e^{2t} & 6te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}}$$

II. Matrix Exponential Solutions:

IVP: $\underline{x}' = \underline{\underline{A}}\,\underline{x}$ $\qquad \underline{x}(0) = \underline{x}_0$

solution is: $\underline{x}(t) = e^{\underline{\underline{A}}t}\underline{x}_0$

Ex: Solve the IVP:

$$\underline{x}' = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix}\underline{x} \qquad \underline{x}(0) = \begin{bmatrix} 19 \\ 29 \\ 39 \end{bmatrix}$$

Calculated $e^{\underline{\underline{A}}t}$

$$\underline{x}(t) = e^{\underline{\underline{A}}t}\underline{x}_0 = \begin{bmatrix} e^{2t} & 3te^{2t} & (4t+9t^2)e^{2t} \\ 0 & e^{2t} & 6te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}\begin{bmatrix} 19 \\ 29 \\ 39 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t}[19 + 29(3t) + 39(4t+9t^2)] \\ e^{2t}[0 + 29(1) + 39(6t)] \\ e^{2t}[0 + 0 + 39(1)] \end{bmatrix}$$

$$\underline{x}(t) = \begin{bmatrix} (19 + 243t + 351t^2)e^{2t} \\ (29 + 234t)e^{2t} \\ \cdots \end{bmatrix}$$

$$\boxed{\underline{x}(t) = \begin{bmatrix} (19 + 243t + 351t^2)e^{2t} \\ (29 + 234t)e^{2t} \\ 39e^{2t} \end{bmatrix}}$$

Q: What if $\underline{\underline{A}}$ cannot be reduced to (diagonal + nilpotent)

$$\underline{\underline{A}} = \underline{\underline{D}} + \underline{\underline{B}}$$

Ans: Let's use the eigenvalue method to calculate $e^{\underline{\underline{A}}t}$.

Def: If $\underline{x}_1(t), \underline{x}_2(t), \ldots, \underline{x}_n(t)$ are n linearly independent solutions of $\underline{x}' = \underline{\underline{A}}\,\underline{x}$, then the $n \times n$ matrix

$$\underline{\underline{\Phi}}(t) = \begin{bmatrix} | & | & & | \\ \underline{x}_1 & \underline{x}_2 & \cdots & \underline{x}_n \\ | & | & & | \end{bmatrix}$$

is called the fundamental matrix of the system.

Thm 1: The solution to the matrix problem

$$\underline{x}' = \underline{\underline{A}}\,\underline{x} \qquad \underline{x}(0) = \underline{x}_0$$

is given by $\boxed{\underline{x}(t) = \underline{\underline{\Phi}}(t)\,\underline{\underline{\Phi}}(0)^{-1}\,\underline{x}_0}$

NOTE: Let $\underline{x}(t) = \underline{\underline{\Phi}}(t)\,\underline{\underline{\Phi}}(0)^{-1}\,\underline{x}_0 = e^{\underline{\underline{A}}t}\,\underline{x}_0$

then $\boxed{e^{\underline{\underline{A}}t} = \underline{\underline{\Phi}}(t)\,\underline{\underline{\Phi}}(0)^{-1}}$

then $\boxed{e^{\underline{\underline{A}}t} = \underline{\underline{\Phi}}(t)\,\underline{\underline{\Phi}}(0)^{-1}}$

NOTE: From the eigenvalue method, we know the fundamental solutions of $\underline{x}' = \underline{\underline{A}}\,\underline{x}$

$$\underline{x}_1(t) = e^{\lambda_1 t}\underline{v}^{(1)}, \quad \underline{x}_2(t) = e^{\lambda_2 t}\underline{v}^{(2)}, \quad \ldots$$

use these to write the fundamental matrix.

Ex: $\underline{x}' = \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}\underline{x}$ $\qquad \underline{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\underline{\underline{A}} = \underline{\underline{D}} + \underline{\underline{B}} = \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}}_{\text{Diagonal}} + \underbrace{\begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix}}_{\text{Nilpotent?}} \times$$

This is not a nilpotent matrix

1. Find the fundamental solns from the Eigenvalue Method:

$\lambda_1 = -1 \qquad \underline{v}^{(1)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda_2 = 5 \qquad \underline{v}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

fundamental solns:

$\underline{x}_1 = e^{-t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \underline{x}_2(t) = e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. Fundamental matrix:

$$\underline{\underline{\Phi}}(t) = \begin{bmatrix} e^{-t} & e^{5t} \\ -2e^{-t} & e^{5t} \end{bmatrix}$$

3. Formula:

$$e^{\underline{\underline{A}}t} = \underline{\underline{\Phi}}(t)\,\underline{\underline{\Phi}}(0)^{-1}$$

$$\underline{\underline{\Phi}}(0) = \underline{\underline{\Phi}}(t)\Big|_{t=0} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

__NOTE:__ The inverse of a 2x2 matrix is

$$\underline{\underline{B}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \underline{\underline{B}}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\underline{\underline{\Phi}}(0)^{-1} = \frac{1}{(1)(1)-(1)(-2)}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$e^{\underline{\underline{A}}t} = \begin{bmatrix} e^{-t} & e^{5t} \\ -2e^{-t} & e^{5t} \end{bmatrix}\left\{\frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\right\}$$

$$e^{\underline{\underline{A}}t} = \frac{1}{3}\begin{bmatrix} e^{-t}+2e^{5t} & -e^{-t}+e^{5t} \\ -2e^{-t}+2e^{5t} & 2e^{-t}+e^{5t} \end{bmatrix}$$

4. Find the solution

$$\underline{x}(t) = e^{\underline{\underline{A}}t}\underline{x}_0 = \frac{1}{3}\begin{bmatrix} e^{-t}+2e^{5t} & -e^{-t}+e^{5t} \\ -2e^{-t}+2e^{5t} & 2e^{-t}+e^{5t} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} e^{-t}+2e^{5t} - (-e^{-t}+e^{5t}) \\ -2e^{-t}+2e^{5t} - (2e^{-t}+e^{5t}) \end{bmatrix}$$

$$\boxed{\underline{x}(t) = \frac{1}{3}\begin{bmatrix} 2e^{-t} + e^{5t} \\ -4e^{-t} + e^{5t} \end{bmatrix}}$$

NOTE: The advantage of this approach is that it is faster to solve for the initial conditions $\underline{x}(0) = \underline{x}_0$

Only need to calculate $e^{\underline{A}t}$ once.