☆ Nonhomogeneous Linear Systems

Warm up: Which of the following are methods to solve nonhomogeneous ODEs? (select all that apply)

(a) Undetermined Coefficients
(b) Elimination
(c) Reduction of Order
(d) Variation of Parameters
(e) Separation of Variables

Project # 3 – due Thurs Apr 8

Predator–Prey Eqns

nonlinear system of ODEs

$$x' = x(1-y)$$
$$y' = y(x-1)$$

x – aphids (millions of)
y – lady bugs (millions)

use pplane8 to plot the dynamics of system when: $x(0) = 0.8$, $y(0) = 0.4$

→ pplane8 posted in BS
→ YouTube tutorial for using pplane (BS week12)

Submission:
→ plots
→ answers to questions
✗ NO CODE

I. Nonhomogeneous Systems

$x' = Ax + f(t)$

I. Nonhomogeneous Sys.

$$\underline{x}' = \underline{\underline{A}}\,\underline{x} + \underbrace{\underline{f}(t)}_{\text{nonhomogeneous term}}$$

Recall in 1D:

$$ay'' + by' + cy = f(t)$$

1. Find the general soln: $y_c(t)$
2. Find a particular soln: $y_p(t)$
   a. Undetermined Coeff.
   b. Variation of Parameters
3. Then $y(t) = y_c(t) + y_p(t)$

Same procedure for systems:

1. general soln:
   $$\underline{x}_c(t) = c_1 \underline{x}_1(t) + \dots + c_n \underline{x}_n(t)$$
   → Elimination
   → Eigenvalue Method
2. particular soln: $\underline{x}_p(t)$

II. Undetermined Coefficients

$$\underline{x}' = \underline{\underline{A}}\,\underline{x} + \underline{f}(t)$$

1. Guess Ansatz has similar form to $\underline{f}(t)$
2. Coefficients are now a <u>vector</u>

Ex: $$\underline{x}' = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \underline{x} + \begin{bmatrix} 3 \\ 2t \end{bmatrix}$$

$$\underline{f}(t) = \begin{bmatrix} 3 \\ 2t \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} t$$

$\underline{\text{Ansatz}}$: $\underline{x}_p(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} t = \begin{bmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{bmatrix}$

Plug into ODE:

$$\underline{x}' = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \underline{\underline{A}}\,\underline{x} + \underline{f}(t) =$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{bmatrix} + \begin{bmatrix} 3 \\ 2t \end{bmatrix}$$

$$= \begin{bmatrix} 3a_1 + 3b_1 t + 2a_2 + 2b_2 t \\ 7a_1 + 7b_1 t + 5a_2 + 5b_2 t \end{bmatrix} + \begin{bmatrix} 3 \\ 2t \end{bmatrix}$$

separate like terms

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = t \underbrace{\begin{bmatrix} 3b_1 + 2b_2 \\ 7b_1 + 5b_2 + 2 \end{bmatrix}}_{= \begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \begin{bmatrix} 3a_1 + 2a_2 + 3 \\ 7a_1 + 5a_2 \end{bmatrix}$$

system of 4 eqns + 4 unknowns

$3b_1 + 2b_2 = 0$

$7b_1 + 5b_2 + 2 = 0$

$b_2 = -\frac{3}{2} b_1 = \boxed{-6 = b_2}$

$2 \times \left(7b_1 + 5\left(-\frac{3}{2} b_1\right) = -2\right)$

$14 b_1 - 15 b_1 = -4$

$\boxed{b_1 = 4}$

$b_1 = 3a_1 + 2a_2 + 3$

$b_2 = 7a_1 + 5a_2$

$4 = 3a_1 + 2a_2 + 3$

$-6 = 7a_1 + 5a_2$

$1 = 3a_1 + 2a_2$

$a_2 = \frac{1}{2}(1 - 3a_1)$

$14b_1 - 13b_1$

$\boxed{b_1 = 4}$

$a_2 = \frac{1}{2}(1-3a_1)$

$\left\{-6 = 7a_1 + 5\left[\frac{1}{2}(1-3a_1)\right]\right\} \times 2$

$-12 = 14a_1 + 5 - 15a_1$

$-17 = -a_1 \quad \boxed{a_1 = 17}$

$a_2 = \frac{1}{2}(1-3\cdot 17) = -25$

$$\underline{x}(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} t$$

$$= \begin{bmatrix} 17 \\ -25 \end{bmatrix} + \begin{bmatrix} 4 \\ -6 \end{bmatrix} t = \boxed{\begin{bmatrix} 17+4t \\ -25-6t \end{bmatrix} = \underline{x}(t)}$$

II. Duplicate Expressions

Like in the 1D case, if $\underline{x}_p(t)$ duplicates terms in $\underline{x}_c(t)$, we need to adjust ansatz.

Ex: $\underline{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \underline{x} - \begin{bmatrix} 15 \\ 4 \end{bmatrix} t e^{-2t}$

1. General soln $\rightarrow$ Eigenvalue method

$\lambda_1 = -2 \qquad \underline{v}^{(1)} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$\lambda_2 = 5 \qquad \underline{v}^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\underline{x}_c(t) = c_1 e^{-2t}\begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

2. Undetermined Coeff:

Originally, ansatz would be:

$\cancel{\underline{x}_p(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t e^{-2t} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e^{-2t}}$

Originally, ansatz w...

$$\underline{x}_p(t) = \cancel{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} te^{-2t} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e^{-2t}}$$

duplicates term in the general solution

In 1D, we would multiply by t^s $(s=1)$

In systems, we need all lower polynomial terms

$$t \longrightarrow (a+bt)$$

$$\underline{x}_p(t) = \cancel{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t^2 e^{-2t} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} te^{-2t}}$$

Instead

$$\underline{x}_p(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t^2 e^{-2t} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} te^{-2t} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{-2t}$$

Now $\longrightarrow$ 6 equations + 6 unknowns

Plug into ODE:

$$\underline{x}_p' = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \left(2te^{-2t} + t^2(-2e^{-2t})\right) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \left(e^{-2t} + t(-2e^{-2t})\right) + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \left(-2e^{-2t}\right)$$

$$= \cancel{e^{-2t}} \left\{ t^2 \begin{bmatrix} -2a_1 \\ -2a_2 \end{bmatrix} + t \begin{bmatrix} 2a_1 - 2b_1 \\ 2a_2 - 2b_2 \end{bmatrix} + \begin{bmatrix} b_1 - 2c_1 \\ b_2 - 2c_2 \end{bmatrix} \right\}$$

$$= \underline{\underline{A}}\,\underline{x} + \underline{f}(t)$$

$$= \begin{bmatrix} 4 & 2 \\ \end{bmatrix} \cancel{e^{-2t}} \left\{ t^2 \begin{bmatrix} a_1 \\ \end{bmatrix} + t \begin{bmatrix} b_1 \\ \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right\} - \begin{bmatrix} 15 \\ 4 \end{bmatrix} t\cancel{e^{-2t}}$$

$$= \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \cancel{e^{-2t}} \left\{ t^2 \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + t \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right\} - \begin{bmatrix} 15 \\ 4 \end{bmatrix} t \cancel{e}$$

cancel out e^{-2t} terms

$$LHS = t^2 \begin{bmatrix} 4a_1 + 2a_2 \\ 3a_1 - a_2 \end{bmatrix} + t \begin{bmatrix} 4b_1 + 2b_2 - 15 \\ 3b_1 - a_2 - 4 \end{bmatrix} + \begin{bmatrix} 4c_1 + 2c_2 \\ 3c_1 - c_2 \end{bmatrix}$$

match up similiar terms

$$t^2 : \quad \begin{bmatrix} -2a_1 \\ -2a_2 \end{bmatrix} = \begin{bmatrix} 4a_1 + 2a_2 \\ 3a_1 - a_2 \end{bmatrix}$$

$$t : \quad \begin{bmatrix} 2a_1 - 2b_1 \\ 2a_2 - 2b_2 \end{bmatrix} = \begin{bmatrix} 4b_1 + 2b_2 - 15 \\ 3b_1 - b_2 - 4 \end{bmatrix}$$

$$1 : \quad \begin{bmatrix} b_1 - 2c_1 \\ b_2 - 2c_2 \end{bmatrix} = \begin{bmatrix} 4c_1 + 2c_2 \\ 3c_1 - c_2 \end{bmatrix}$$

6 eqns
6 unknowns

write as matrix eqn

$$\underline{\underline{C}}\, \underline{x} = \underline{y}$$

$\underline{\underline{C}}$ - 6x6

$\underline{y}$ - 6x1

$$\underline{x} = \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{bmatrix}$$

$$6a_1 + 2a_2 = 0$$
$$3a_1 + a_2 = 0$$
$$2a_1 - 6b_1 - 2b_2 = -15$$
$$2a_2 - 3b_1 - b_2 = -4$$
$$b_1 - 6c_1 - 2c_2 = 0$$
$$b_2 - 3c_1 - c_2 = 0$$

Becomes

$$\underbrace{\begin{bmatrix} 6 & 2 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & -6 & -2 & 0 & 0 \\ 0 & 2 & -3 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -6 & -2 \\ 0 & 0 & 0 & 1 & -3 & -1 \end{bmatrix}}_{\underline{\underline{C}}} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ -15 \\ -4 \\ 0 \\ 0 \end{bmatrix}}_{\underline{y}}$$

plug into a computer algebra system
MATLAB, Mathematica, etc.

$a_1 = \frac{-1}{2}$, $a_2 = \frac{3}{2}$, $b_1 = 2$, $b_2 = 1$, $c_1 = \frac{1}{3}$, $c_2 = 0$

so $$\boxed{\underline{x}_p(t) = \begin{bmatrix} -1/2 \\ 3/2 \end{bmatrix} t^2 e^{-2t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} t e^{-2t} + \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} e^{-2t}}$$

<u>NEXT TIME</u> – Variation of Parameters