☆ Variation of Parameters

Warm up: Write down the definition of the fundamental matrix $\underline{\underline{\Phi}}(t)$ of the linear system $\underline{x}' = \underline{\underline{A}}\,\underline{x}$ ($\underline{\underline{A}}$ $n \times n$)

Ans: If $\underline{x}_1(t), \ldots, \underline{x}_n(t)$ are the fundamental soln s

$$\underline{\underline{\Phi}}(t) = \begin{bmatrix} | & | & & | \\ \underline{x}_1(t) & \underline{x}_2(t) & \cdots & \underline{x}_n(t) \\ | & | & & | \end{bmatrix} \quad (n \times n)$$

I. Variation of Parameters:

Nonhomogeneous linear systems

$$\underline{x}' = \underbrace{\underline{\underline{P}}(t)}\,\underline{x} + \underline{f}(t)$$

coefficent matrix $\underline{\underline{P}}(t)$ can be a fcn of t (as opposed to const. $\underline{\underline{A}}$)

Given the general soln of $\underline{x}' = \underline{\underline{P}}(t)\underline{x}$

$$\underline{x}_c(t) = c_1\underline{x}_1(t) + c_2\underline{x}_2(t) + \ldots + c_n\underline{x}_n(t)$$

NOTE: we can write $\underline{x}_c(t)$ in terms of $\underline{\underline{\Phi}}(t)$

$$\boxed{\underline{x}_c(t) = \underline{\underline{\Phi}}(t)\,\underline{c}} = \begin{bmatrix} | & | & & | \\ \underline{x}_1 & \underline{x}_2 & \cdots & \underline{x}_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = c_1\underline{x}_1(t) + c_2\underline{x}_2(t) + \ldots + c_n\underline{x}_n(t)$$

$$\underline{x}_c(t) = \underline{\underline{\Phi}}(t)\,\underline{c}$$

1D

$y_c = c_1 y_1 + c_2 y_2$

Ansatz:

$y_p = u_1(t)y_1(t) + u_2(t)y_2(t)$

GOAL: $u_1(t)$ and $u_2(t)$

Variation of Parameters:

Ansatz: $\underline{x}_p(t) = \underline{\underline{\Phi}}(t)\,\underbrace{\underline{u}(t)}$

undetermined vector

undetermined vector

$$\underline{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix}$$

GOAL: $u_1(t)$ and $u_2(t)$

Plug into the ODE

$$\underline{x}_p' = \underline{\underline{P}}(t)\,\underline{x}_p + \underline{f}(t)$$

$$\underbrace{\underline{\underline{\Phi}}'(t)\,\underline{u}(t) + \underline{\underline{\Phi}}(t)\,\underline{u}'(t)}_{\text{product rule for derivs}} = \underline{\underline{P}}(t)\,\underline{\underline{\Phi}}(t)\,\underline{u}(t) + \underline{f}(t)$$

$$\underline{\underline{\Phi}}'(t) = \begin{bmatrix} | & | & & | \\ \underline{x}_1'(t) & \underline{x}_2'(t) & \cdots & \underline{x}_n'(t) \\ | & | & & | \end{bmatrix}$$

but, we know that $\underline{x}_i$ solves $\underline{x}_i' = \underline{\underline{P}}(t)\,\underline{x}_i$

$$= \begin{bmatrix} | & | & & | \\ \underline{\underline{P}}(t)\underline{x}_1 & \underline{\underline{P}}(t)\underline{x}_2 & \cdots & \underline{\underline{P}}(t)\underline{x}_n \\ | & | & & | \end{bmatrix}$$

$$\underline{\underline{\Phi}}'(t) = \underline{\underline{P}}(t)\,\underline{\underline{\Phi}}(t)$$

$$\cancel{\underline{\underline{P}}(t)\,\underline{\underline{\Phi}}(t)\,\underline{u}(t)} + \underline{\underline{\Phi}}(t)\,\underline{u}'(t) = \cancel{\underline{\underline{P}}(t)\,\underline{\underline{\Phi}}(t)\,\underline{u}(t)} + \underline{f}(t)$$

$$\underline{\underline{\Phi}}(t)\,\underline{u}'(t) = \underline{f}(t)$$

$$\int \underline{u}'(t) = \int \underline{\underline{\Phi}}^{-1}(t)\,\underline{f}(t)$$

$$\underline{u}(t) = \int \underline{\underline{\Phi}}^{-1}(t)\,\underline{f}(t)\,dt$$

Then, our solution

$$\boxed{\underline{x}_p(t) = \underline{\underline{\Phi}}(t)\underline{u}(t) = \underline{\underline{\Phi}}(t)\int \underline{\underline{\Phi}}^{-1}(s)\,\underline{f}(s)\,ds}$$

Case: $\underline{\underline{P}}(t) = \underline{\underline{A}}$

we can use the exponential matrix

$$e^{\underline{\underline{A}}t} = \underline{\underline{\Phi}}(t)\,\underline{\underline{\Phi}}(0)^{-1}$$

NOTE: $\left[e^{\underline{\underline{A}}t}\right]^{-1} = e^{-\underline{\underline{A}}t}$

NOTE:

$[\underline{\underline{A}}\,\underline{\underline{B}}]^{-1} = \underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1}$

$(\underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1})(\underline{\underline{A}}\,\underline{\underline{B}})$

$= \underline{\underline{B}}^{-1}(\underline{\underline{A}}^{-1}\underline{\underline{A}})\,\underline{\underline{B}}$

$= \underline{\underline{B}}^{-1}\underline{\underline{I}}\,\underline{\underline{B}}$

$= \underline{\underline{B}}^{-1}\underline{\underline{B}} = \underline{\underline{I}}$

we can write

$$\underline{\underline{\Phi}}(t) = e^{\underline{\underline{A}}t}\,\underline{\underline{\Phi}}(0)$$

$$\underline{\underline{\Phi}}^{-1}(t) = \left[e^{\underline{\underline{A}}t}\,\underline{\underline{\Phi}}(0)\right]^{-1} = \underline{\underline{\Phi}}(0)^{-1}\,e^{-\underline{\underline{A}}t}$$

So $\underline{x}_p(t)$ becomes

$$\underline{x}_p(t) = \underline{\underline{\Phi}}(t)\int \underline{\underline{\Phi}}^{-1}(s)\,\underline{f}(s)\,ds$$

$$= e^{\underline{\underline{A}}t}\,\underline{\underline{\Phi}}(0)\int \underbrace{\underline{\underline{\Phi}}(0)^{-1}}_{\text{constant}}\,e^{-\underline{\underline{A}}s}\,\underline{f}(s)\,ds$$

constant — move outside integral

$$= e^{\underline{\underline{A}}t}\,\underbrace{\underline{\underline{\Phi}}(0)\,\underline{\underline{\Phi}}(0)^{-1}}_{=\underline{\underline{I}}}\int e^{-\underline{\underline{A}}s}\,\underline{f}(s)\,ds$$

$e^{\underline{\underline{A}}t}\,\underline{\underline{I}} = e^{\underline{\underline{A}}t}$

So $$\boxed{\underline{x}_p(t) = e^{\underline{\underline{A}}t}\int e^{-\underline{\underline{A}}s}\,\underline{f}(s)\,ds}$$

IVP: $\underline{x}' = \underline{\underline{A}}\,\underline{x} + \underline{f}(t)$ initial cond $\underline{x}(0) = \underline{x}_0$

then $\underline{x}_c(t) = e^{\underline{\underline{A}}t}\underline{x}_0$

full soln:

$$\underline{x}(t) = \underline{x}_c + \underline{x}_p$$

$$= e^{\underline{\underline{A}}t}\underline{x}_0 + e^{\underline{\underline{A}}t}\int_0^t e^{-\underline{\underline{A}}s}\underline{f}(s)\,ds$$

$$\boxed{\underline{x}(t) = e^{\underline{\underline{A}}t}\left[\underline{x}_0 + \int_0^t e^{-\underline{\underline{A}}s}\underline{f}(s)\,ds\right]}$$

Ex: Solve the IVP:

$$\underline{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\underline{x} - \begin{bmatrix} 15 \\ 4 \end{bmatrix} te^{-2t} \qquad \underline{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

we solved this last lecture w/ Undet. Coeff
- 6 eqns + 6 unknowns

Variation of Parameters:

1. Solve the homogeneous eqn: $\longrightarrow$ Eigenvalue Method

$$\lambda_1 = -2 \qquad \underline{v}^{(1)} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\lambda_2 = 5 \qquad \underline{v}^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

2. Find $e^{\underline{\underline{A}}t} = \underline{\underline{\Phi}}(t)\,\underline{\underline{\Phi}}^{-1}(0)$

$$\underline{\underline{\Phi}}(t) = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \qquad \underline{\underline{\Phi}}(0) = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$

$$\underline{\Phi}(0)^{-1} = \frac{1}{1+6}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

$$e^{\underline{A}t} = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \frac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

$$e^{\underline{A}t} = \frac{1}{7}\begin{bmatrix} e^{-2t}+6e^{5t} & -2e^{-2t}+2e^{5t} \\ -3e^{-2t}+3e^{5t} & 6e^{-2t}+e^{5t} \end{bmatrix}$$

3. Variation of Parameters formula

$$\underline{x}(t) = e^{\underline{A}t}\underbrace{\left[\underline{x}_0 + \int_0^t e^{-\underline{A}s}\underline{f}(s)\,ds\right]}_{\underline{w}}$$

$$e^{-\underline{A}s} = e^{\underline{A}t}\,(t \longmapsto -s)$$

$$\underline{w} = \underline{x}_0 + \int_0^t e^{-\underline{A}s}\underline{f}(s)\,ds$$

$$= \begin{bmatrix} 7 \\ 3 \end{bmatrix} + \int_0^t \frac{1}{7}\begin{bmatrix} e^{2s}+6e^{-5s} & -2e^{2s}+2e^{-5s} \\ -3e^{2s}+3e^{-5s} & 6e^{2s}+e^{-5s} \end{bmatrix}\begin{bmatrix} -15se^{-2s} \\ -4se^{-2s} \end{bmatrix} ds$$

$$= \begin{bmatrix} 7 \\ 3 \end{bmatrix} + \int_0^t \begin{bmatrix} -s - 14se^{-7s} \\ 3s - 7se^{-7s} \end{bmatrix} ds$$

$$= \begin{bmatrix} 7 \\ 3 \end{bmatrix} + \begin{bmatrix} \int_0^t -s - 14se^{-7s}\,ds \\ \int_0^t 3s - 7se^{-7s}\,ds \end{bmatrix}$$

Integration by parts

$$= \begin{bmatrix} 7 \\ 3 \end{bmatrix} + \frac{1}{14}\begin{bmatrix} -4 - 7t^2 + 4e^{-7t} + 28te^{-7t} \\ -2 + 21t^2 + 2e^{-7t} + 14te^{-7t} \end{bmatrix}$$

$$\underline{w} = \frac{1}{14}\begin{bmatrix} 94 - 7t^2 + 4e^{-7t} + 28te^{-7t} \\ 40 + 21t^2 + 2e^{-7t} + 14te^{-7t} \end{bmatrix}$$

$$\underline{x}(t) = e^{\underline{\underline{A}}t}\, \underline{w}$$

$$= \frac{1}{7}\begin{bmatrix} e^{-2t} + 6e^{5t} & -2e^{-2t} + 2e^{5t} \\ -3e^{-2t} + 3e^{5t} & 6e^{-2t} + e^{5t} \end{bmatrix} \underline{w}$$

matrix mult:

$$\boxed{\underline{x}(t) = \frac{1}{14}\begin{bmatrix} (6 + 28t - 7t^2)e^{-2t} + 92e^{5t} \\ (-4 + 14t + 21t^2)e^{-2t} + 46e^{5t} \end{bmatrix}}$$

FOR THE HW:

can use computer algebra system to do these matrix multiplication.

Ex: Sec 5.7 #17

$$\underline{x}' = \underline{\underline{A}}\,\underline{x} + \underline{f}(t) \qquad \underline{x}(0) = \underline{x}_0$$

Gives:

$$\underline{\underline{A}} = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix} \qquad \underline{f}(t) = \begin{bmatrix} 60 \\ 90 \end{bmatrix} \qquad \underline{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$e^{\underline{\underline{A}}t} = \frac{1}{6}\begin{bmatrix} -e^{-t} + 7e^{5t} & 7e^{-t} - 7e^{5t} \\ -e^{-t} + e^{5t} & 7e^{-t} - e^{5t} \end{bmatrix}$$

use variation of parameters to find $\underline{x}(t)$

$$\underline{x}(t) = e^{\underline{\underline{A}}t}\left[\underline{x}_0 + \int_0^t e^{-\underline{\underline{A}}s}\, \underline{f}(s)\, ds\right]$$

$$= e^{\underline{\underline{A}}t}\left(\cancel{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}^{\,0} + \int_0^t \frac{1}{6}\begin{bmatrix} -e^{s} + 7e^{-5s} & 7e^{s} - 7e^{-5s} \\ -e^{s} + e^{-5s} & 7e^{s} - e^{-5s} \end{bmatrix}\begin{bmatrix} 60 \\ 90 \end{bmatrix} ds\right)$$

$$= e^{\underline{\underline{A}}t} \int_0^t \frac{1}{6} \begin{bmatrix} 570e^{s} - 210e^{-5s} \\ 570e^{s} - 30e^{-5s} \end{bmatrix} ds$$

$$= e^{\underline{\underline{A}}t} \int_0^t \begin{bmatrix} 115e^{s} - 35e^{-5s} \\ 115e^{s} - 5e^{-5s} \end{bmatrix} ds$$

$$= e^{\underline{\underline{A}}t} \begin{bmatrix} 115e^{s} + 7e^{-5s} \\ 115e^{s} + e^{-5s} \end{bmatrix} \Bigg|_0^t$$

$$= e^{\underline{\underline{A}}t} \begin{bmatrix} 115e^{t} + 7e^{-5t} - 122 \\ 115e^{t} + e^{-5t} - 116 \end{bmatrix}$$

matrix multiplication $\rightarrow$ computer algebra system.