✩ Laplace Transforms

Warm up: Use Euler's formula $e^{it} = \cos(t) + i\sin(t)$ To write $\cos(t)$ in terms of complex exponentials

Ans: $e^{-it} = \cos(t) - i\sin(t)$

so $\cos(t) = \frac{1}{2}\left[e^{it} + e^{-it}\right]$

I. Laplace Transforms

→ powerful tool to solve linear ODE with variable coeff

$$a(t)y'' + b(t)y' + c(t)y = f(t)$$

Def: The Laplace Transform of a function $f(t)$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty \underbrace{e^{-st}}_{\text{"kernel" of the transform}} f(t)\,dt$$

Recall: Linear ODE ⟶ solns are expontials so the kernel e^{-st} is good a choice

Ex: Find the Laplace Transform (L.T.) of $f(t) = 1$

indefinite integral

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \int_0^\infty e^{-st}\,dt$$

$$= \lim_{b\to\infty}\int_0^b e^{-st}\,dt = \lim_{b\to\infty}\left[\frac{e^{-st}}{-s}\right]_0^b$$

$$= \lim_{b\to\infty}\left[\underbrace{\frac{e^{-sb}}{-s}}_{\to 0 \text{ if } s>0} - \frac{1}{-s}\right] = \frac{1}{s}$$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s} \quad \text{if } s>0}$$

$\xrightarrow{\ \ } 0$ if $s>0$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s} \quad \text{if } s>0}$$

Q: Find the L.T. of $f(t) = 3$

a) $\mathcal{L}\{3\} = \frac{1}{s}$ $\quad$ b) $\boxed{\mathcal{L}\{3\} = \frac{3}{s}}$ $\quad$ c) $\mathcal{L}\{3\} = \frac{1}{s^3}$

$$\mathcal{L}\{3\} = \int_0^\infty e^{-st}(3)\,dt = 3\underbrace{\int_0^\infty e^{-st}\,dt} = 3\mathcal{L}\{1\}$$

$$= 3\left(\frac{1}{s}\right) = \frac{3}{s}$$

$$\boxed{\mathcal{L}\{a f(t)\} = a\mathcal{L}\{f(t)\} \quad \text{where } a \text{ is a constant}}$$

Ex. Find the L.T. of $f(t) = e^{at}$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}e^{at}\,dt = \lim_{b\to\infty}\int_0^b e^{-(s-a)t}\,dt$$

$$= \lim_{b\to\infty}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^b = \lim_{b\to\infty}\left[\frac{e^{-(s-a)b}}{-(b-a)} - \frac{1}{-(s-a)}\right]$$

$\xrightarrow{\ \ } 0$ if $s>a$

$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad \text{if } s>a}$$

II. Linearity:

Ex: Find the L.T. of $f(t) = 1 + e^{-3t}$

$$\mathcal{L}\{1 + e^{-3t}\} = \int_0^\infty e^{-st}(1 + e^{-3t})\,dt$$

because the integral is linear

$$= \int_0^\infty e^{-st}\,dt + \int_0^\infty e^{-st}e^{-3t}\,dt$$

$$= \int_0^\infty e^{-st}\,dt + \int_0 e\ e\ dt$$

$$= \mathcal{L}\{1\} + \mathcal{L}\{e^{-3t}\} = \boxed{\frac{1}{s} + \frac{1}{s+3}}$$

The Laplace Transform satisfies

$$\mathcal{L}\{a f(t)\} = a\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f(t)+g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$$

we say that the L.T. is linear

NOTE: This is NOT true for multiplication of fcns

$$\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$$

III. More Transforms:

Ex: Find the L.T. of $f(t) = \cos(t)$

$$\mathcal{L}\{\cos(t)\} = \int_0^\infty e^{-st}\cos(t)\,dt$$ ← difficult integral

Instead, expand $\cos(t)$ in terms of Euler's formula

$$\cos(t) = \frac{1}{2}\left[e^{it} + e^{-it}\right]$$

$$\mathcal{L}\{\cos(t)\} = \mathcal{L}\left\{\frac{1}{2}\left[e^{it} + e^{-it}\right]\right\}$$

$$= \frac{1}{2}\mathcal{L}\{e^{it}\} + \frac{1}{2}\mathcal{L}\{e^{-it}\}$$

$s > Re(i) = 0$

$s > 0$

$$= \frac{1}{2}\left(\frac{1}{s-i}\right) + \frac{1}{2}\left(\frac{1}{s+i}\right)$$

find a common denominator

$-\frac{1}{2}(s-i) \qquad 2(s+i)$

find a common denominator

$$= \frac{1}{2}\,\frac{(s+i)+(s-i)}{(s-i)(s+i)} = \frac{\frac{1}{2}(2s)}{s^2+1}$$

$$\boxed{\mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1}}$$

Ex: Find the L.T. of $f(t) = 5t$

$$\mathcal{L}\{5t\} = \int_0^{\infty} e^{-st}(5t)\,dt = 5\int_0^{\infty} te^{-st}\,dt$$

Integration by Parts:

$u = t \qquad dv = e^{-st}dt$

$du = dt \qquad v = \frac{e^{-st}}{-s}$

$$= 5\lim_{b\to\infty}\left[\left(\frac{te^{-st}}{-s}\right)\Big|_0^b - \int_0^b \frac{e^{-st}}{-s}\,dt\right]$$

$$= 5\lim_{b\to\infty}\left[\left(\frac{te^{-st}}{-s} - \frac{e^{-st}}{(-s)^2}\right)\Big|_0^b\right]$$

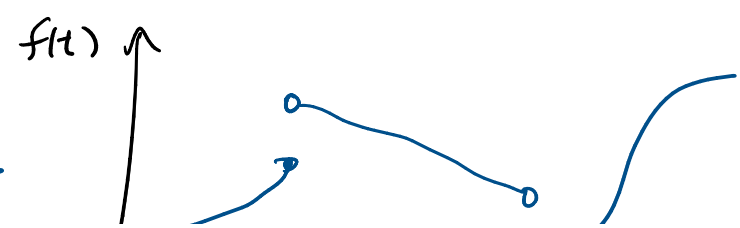
$$= 5\lim_{b\to\infty}\left[\underbrace{\frac{be^{-sb}}{-s} - \frac{e^{-sb}}{s^2}}_{\to 0 \text{ if } s>0} + \left(-0 + \frac{1}{s^2}\right)\right]$$

$$= 5\left(\frac{1}{s^2}\right)$$

so $\boxed{\mathcal{L}\{5t\} = \frac{5}{s^2}}$

IV. Piecewise Continuous Functions

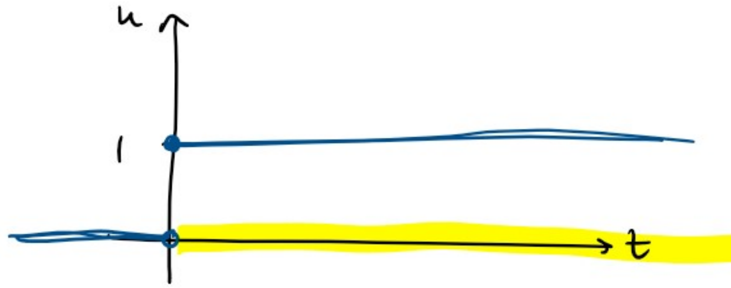
function $f(t)$ has finitely many jumps



function $f(t)$ ...
finitely many jumps

Ex: $u(t) = \begin{cases} 0 & t<0 \\ 1 & t \geq 0 \end{cases}$

unit step function

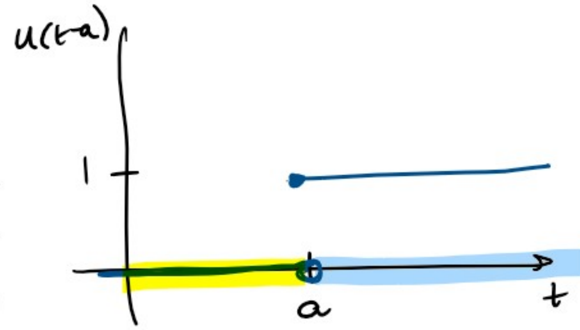


$$\mathcal{L}\{u(t)\} = \int_0^\infty e^{-st} u(t)\,dt = \int_0^\infty e^{-st}(1)\,dt$$

← because $u(t)=1$ when $0 \leq t < \infty$

$$= \mathcal{L}\{1\} = \frac{1}{s}$$

so $\boxed{\mathcal{L}\{u(t)\} = \frac{1}{s}}$



Ex: $u_a(t) = u(t-a) = \begin{cases} 0 & t<a \\ 1 & t \geq a \end{cases}$

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\,dt$$

$$= \int_0^a e^{-st} \underbrace{u(t-a)}_{=0}\,dt + \int_a^\infty e^{-st} \underbrace{u(t-a)}_{=1}\,dt$$

$$= \int_0^a \cancel{0\,dt} + \int_a^\infty e^{-st}\,dt$$

$$= \lim_{b\to\infty} \int_a^b e^{-st}\,dt = \lim_{b\to\infty}\left[\frac{e^{-st}}{-s}\right]_a^b$$

$$= \lim_{b\to\infty}\left[\underbrace{\frac{e^{-sb}}{-s}}_{\to 0} - \frac{e^{-sa}}{-s}\right] = \frac{e^{-as}}{s}$$

if $s>0$

e^{-as}

$b \to \infty \; \Big[\; -s \qquad -s \; \Big] \qquad s$

$\xrightarrow[\text{if } s>0]{} 0$

$$\boxed{\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s} \quad \text{if } s>0}$$

V Inverse Laplace Transform:

If $F(s) = \mathcal{L}\{f(t)\}$, then we call $f(t)$ the inverse Laplace Transform of $F(s)$

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Ex: $\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$

$\mathcal{L}^{-1}\left\{\frac{5}{s^2}\right\} = 5t$

$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos(t)$

In Practice, use a Table of Laplace Transforms

$f(t)$	$F(s)$	
1	$\frac{1}{s}$	$s>0$
t	$\frac{1}{s^2}$	$s>0$
t^n $\left(\begin{matrix} n \geq 0 \\ \text{integer} \end{matrix}\right)$	$\frac{n!}{s^{n+1}}$	$s>0$
e^{at}	$\frac{1}{s-a}$	$s>a$

$\cos(kt)$	$\dfrac{s}{s^2+k^2}$	$s>0$
$\sin(kt)$	$\dfrac{k}{s^2+k^2}$	$s>0$
$u(t-a)$	$\dfrac{e^{-as}}{s}$	$s>0$, $a>0$

$2\sin^2\theta = 1-\cos(2\theta)$

$\theta = 3t$

Ex: $\mathcal{L}\{3e^{2t} + 2\sin^2(3t)\}$

$$= \mathcal{L}\{3e^{2t} + 1 - \cos(6t)\}$$

linearity

$$= 3\mathcal{L}\{e^{2t}\} + \mathcal{L}\{1\} - \mathcal{L}\{\cos(6t)\}$$

$$= 3\left(\frac{1}{s-2}\right) + \left(\frac{1}{s}\right) - \left(\frac{s}{s^2+36}\right)$$

find a common denom.

$$= \boxed{\frac{3s^3 + 144s - 72}{s(s-2)(s^2+36)}}$$