☆ Transformation of Initial Value Problems

Warm up: Write down the definition of the Laplace Transform of $f(t)$?

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

I. IVPs:

Consider the IVP

$$\begin{cases} ax'' + bx' + cx = f(t) \\ x(0) = x_0 \qquad x'(0) = x_0' \end{cases}$$

GOAL: Solve using Laplace Transforms (L.T.)

Procedure:

1. Take L.T. of both sides

$$\mathcal{L}\{ax'' + bx' + cx\} = \mathcal{L}\{f(t)\}$$

2. Since the L.T. is linear

$$a\mathcal{L}\{x''\} + b\mathcal{L}\{x'\} + c\mathcal{L}\{x\} = \mathcal{L}\{f(t)\}$$

Q: What is the L.T. of the derivative of a function $\mathcal{L}\{x'(t)\}$?

Integrate by Parts

$du = x'(t)dt \qquad v = e^{-st}$

$u = x(t) \qquad dv = -se^{-st}dt$

$$\mathcal{L}\{x'(t)\} = \int_0^\infty e^{-st} x'(t)\,dt$$

$$= \lim_{b\to\infty}\left[\left(x(t)e^{-st}\right)_0^b - \int_0^b -se^{-st} x(t)\,dt \right]$$

$$= \lim \left[\left(x(b)e^{-sb} - x(0)\right) + s\int_0^b e^{-st} x(t)\,dt \right]$$

$$= \lim_{b\to\infty}\left[\left(\underbrace{x(b)e^{-sb}}_{\to 0 \text{ if } s>0} - x(0)\right) + s\underbrace{\int_0^b e^{-st}x(t)\,dt}_{\mathcal{L}\{x(t)\}}\right]$$

$$= -x(0) + s\mathcal{L}\{x(t)\}$$

$X(s) = \mathcal{L}\{x(t)\}$

$$\boxed{\mathcal{L}\{x'(t)\} = sX(s) - x(0) \quad \text{if } s>0}$$

NOTE: we will use lower case letters for functions of t $\quad x(t),\ f(t)$

we will use upper case letters for their Laplace transforms $\quad X(s),\ F(s)$

$$X(s) = \mathcal{L}\{x(t)\}, \quad F(s) = \mathcal{L}\{f(t)\}$$

Repeat for higher order derivatives

$$\mathcal{L}\{x''(t)\} = s\mathcal{L}\{x'(t)\} - x'(0)$$

$$= s[sX(s) - x(0)] - x'(0)$$

$$= s^2X(s) - sx(0) - x'(0)$$

$$\mathcal{L}\{x'''(t)\} = s^3X(s) - s^2x(0) - sx'(0) - x''(0)$$

and so on.

Ex: Apply L.T. to an IVP

$$x'' - x' - 6x = 0 \qquad \underline{x(0)=2},\ \underline{x'(0)=-1}$$

$x'' - x - \dots$

1. Take L.T. of both sides

$$\mathcal{L}\{x''\} - \mathcal{L}\{x'\} - 6\,\mathcal{L}\{x\} = 0$$

$$\left[s^2 X(s) - s\underline{x(0)} - \underline{x'(0)}\right] - \left[s X(s) - \underline{x(0)}\right] - 6X(s) = 0$$

$$\left[s^2 X(s) - 2s + 1\right] + \left[-s X(s) + 2\right] - 6X(s) = 0$$

2. Solve for $X(s)$

$$\left[s^2 - s - 6\right] X(s) - 2s + 1 + 2 = 0$$

$$\left[s^2 - s - 6\right] X(s) = 2s - 3$$

$$X(s) = \frac{2s-3}{s^2-s-6} = \underbrace{\frac{2s-3}{(s-3)(s+2)}}$$

Want to take the inverse L.T. of this!
1st rewrite in a more convenient form

3. Expand using the method of partial fractions

$$X(s) = \frac{2s-3}{(s-3)(s+2)} = \underbrace{\frac{A}{s-3}} + \underbrace{\frac{B}{s+2}}$$

this looks like $A\,\mathcal{L}\{e^{3t}\} = \dfrac{A}{s-3}$

looks like $B\,\mathcal{L}\{e^{-2t}\} = \dfrac{B}{s+2}$

Need to find A and B

multiply both sides by $(s-3)(s+2)$

$A(s+2) + B(s-3)$

multiply both sides by

$$2s - 3 = A(s+2) + B(s-3)$$

collect like terms

$$2s - 3 = (A+B)s + (2A - 3B)$$

$$2 = A + B$$

$$A = 2 - B$$

$$= \frac{2\cdot 5}{5} - \frac{7}{5}$$

$$\boxed{A = \frac{3}{5}}$$

$$-3 = 2A - 3B$$

$$-3 = 2(2-B) - 3B$$

$$= 4 - 2B - 3B = 4 - 5B$$

$$-7 = -5B \qquad \boxed{B = \frac{7}{5}}$$

$$X(s) = \frac{3}{5}\left(\frac{1}{s-3}\right) + \frac{7}{5}\left(\frac{1}{s+2}\right)$$

4. Now take the inverse L.T.

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{7}{5}\mathcal{L}\left\{\frac{1}{s+2}\right\}$$

$$\boxed{x(t) = \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}}$$

✩ Laplace Transform Procedure

variable t | variable s

ODE for $x(t)$ → $\mathcal{L}$ → algebraic eqn for $X(s)$

↓

Solve using algebra.

Solve using algebra, partial fractions

$$\boxed{x(t)} \longleftarrow \mathcal{L}^{-1} \longleftarrow \boxed{X(s)}$$

The Laplace Transform and inverse L.T. help us move back and forth between the t and s variables

The L.T. transforms the ODE in t into an algebraic eqn in s.

II. Transform Theorems:

Before we showed that

Thm 1: (Transforms of Derivatives)

$$\mathcal{L}\{x'(t)\} = sX(s) - x(0)$$

Similarly, we have the following rule for integrals

Thm 2: (Transforms of Integrals)

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s}$$

and conversely

and converging

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau$$

We can use Thm 2 to help us find inverse L.T.

Ex: Find the inverse L.T. of $G(s) = \frac{1}{s(s-3)}$

Thm 2: $\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau$

Need to find $F(s)$ and $f(t)$

$$G(s) = \frac{1}{s(s-3)} = \frac{F(s)}{s} \Rightarrow F(s) = \frac{1}{s-3}$$

then $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}$ (look up in Table of L.T.)

$$\text{so } \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} \overset{\text{Thm 2}}{=} \int_0^t f(\tau)\,d\tau$$

$$= \int_0^t e^{3\tau}\,d\tau = \left[\frac{e^{3\tau}}{3}\right]_0^t$$

$$= \frac{1}{3}\left[e^{3t} - 1\right]$$

so $\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s(s-3)}\right\} = \frac{1}{3}\left[e^{3t} - 1\right]}$

III. Systems of ODEs

Ex: $\begin{cases} x' = 2x + y \\ y' = \ldots + 2y \end{cases}$ $\quad x(0) = 1$, $y(0) = -2$

Ex: $\begin{cases} x' = 2x + y \\ y' = 6x + 3y \end{cases}$ $\quad x(0) = 1$, $\quad y(0) = -2$

1. Take the L.T. of both sides

$\mathcal{L}\{x'\} = 2\mathcal{L}\{x\} + \mathcal{L}\{y\} \rightarrow sX(s) - 1 = 2X(s) + Y(s)$

$\mathcal{L}\{y'\} = 6\mathcal{L}\{x\} + 3\mathcal{L}\{y\} \rightarrow sY(s) + 2 = 6X(s) + 3Y(s)$

simplify: $\rightarrow \begin{cases} (s-2)X - Y = 1 \\ -6X + (s-3)Y = -2 \end{cases}$

2. Solve for $X(s)$ and $Y(s)$

$\rightarrow \quad Y = (s-2)X - 1$

plug into 2nd eqn $\quad -6X + (s-3)\left[(s-2)X - 1\right] = -2$

$\left[(s-3)(s-2) - 6\right]X - (s-3) = -2$

$\left[s^2 - 3s - 2s \cancel{+6} \cancel{-6}\right]X = -2 + s - 3 = s - 5$

$s(s-5)X = s - 5$

$X(s) = \frac{\cancel{s-5}}{s\cancel{(s-5)}} = \boxed{\frac{1}{s} = X(s)}$

$Y = (s-2)\left[\frac{1}{s}\right] - 1 = \frac{s-2-s}{s} = \boxed{\frac{-2}{s} = Y(s)}$

3. Take the inverse L.T.

$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$y(t) = \mathcal{L}^{-1}\left\{-\frac{2}{s}\right\} = -2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = -2$

$x(t) = \mathcal{L} \{ s \}$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{-2}{s} \right\} = -2\,\mathcal{L}\left\{ \frac{1}{s} \right\} = -2$$

$$\boxed{x(t) = 1 \qquad y(t) = -2}$$