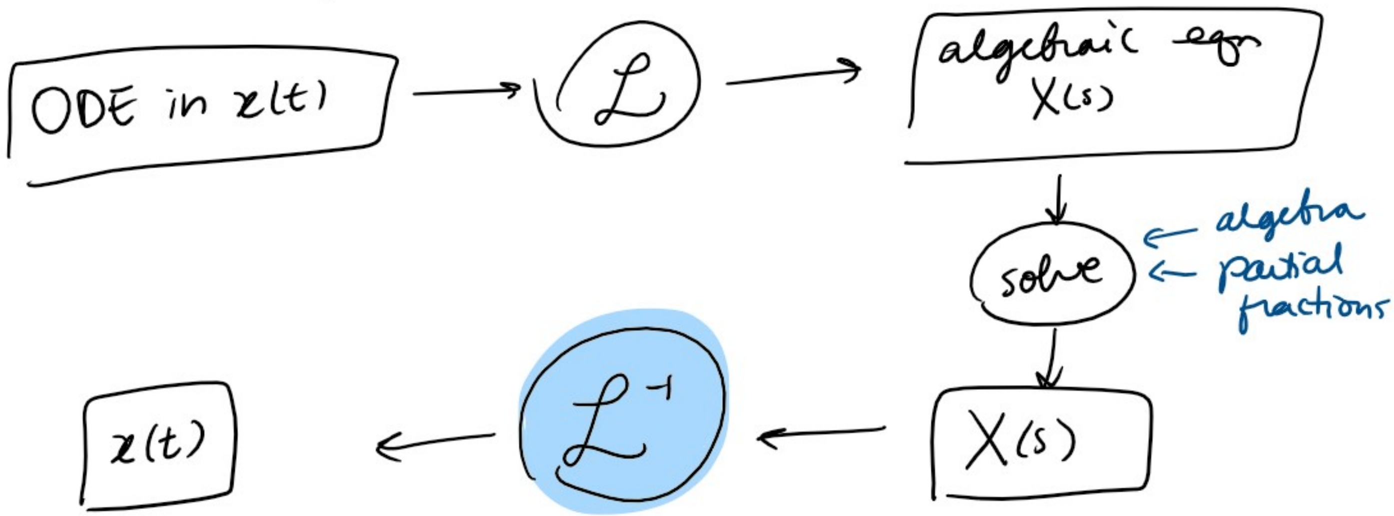
☆ Translation and Partial Fractions

Warm up: Fill in the Table of L.T. below

$f(t)$	$F(s)$	
e^{at}	$\frac{1}{s-a}$	$s>a$
t^n ($n\geq 0$ integer)	$\frac{n!}{s^{n+1}}$	$s>0$
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$s>0$
$\sin(kt)$	$\frac{k}{s^2+k^2}$	$s>0$

Recall, our procedure for solving ODEs using L.T.



KEY STEP: arrange $X(s)$ so that we can take $\mathcal{L}^{-1}\{X(s)\} = x(t)$

I. Translation:

This property helps us take $\mathcal{L}^{-1}\{X(s)\}$

Recall $F(s)=\mathcal{L}\{f(t)\}$

This property helps us find $\mathcal{L}^{-1}\{\ldots\}$

<u>Thm</u> (Translation on the s-axis)

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

and conversely

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$$

Recall $F(s) = \mathcal{L}\{f(t)\}$

t	s
multiply by e^{at}	then $s \longrightarrow s-a$

$$e^{at} e^{-st} = e^{-(s-a)t}$$

<u>Ex</u>: Find the L.T. of $g(t) = e^{3t}\cos(\pi t)$

$$\mathcal{L}\{e^{3t}\cos(\pi t)\} \overset{\text{Thm}}{=} F(s-3)$$

Here $f(t) = \cos(\pi t) \longrightarrow F(s) = \dfrac{s}{s^2+\pi^2}$

$$\mathcal{L}\{g(t)\} = F(s-3) = \boxed{\frac{s-3}{(s-3)^2+\pi^2}}$$

Table of Laplace Transforms

$f(t)$	$F(s)$	
$e^{at} t^n$	$\dfrac{n!}{(s-a)^{n+1}}$	$s>a$
$e^{at}\cos(kt)$	$\dfrac{s-a}{(s-a)^2+k^2}$	$s>a$
$e^{at}\sin(kt)$	$\dfrac{k}{(s-a)^2+k^2}$	$s>a$

<u>Ex</u>: Find the inverse L.T. of $G(s) = \dfrac{2s+3}{s^2+2s+25}$

$s^2+2s+5 = (s+1)^2+4$

Ex: Find

Notice that: $s^2+2s+5 = (s+1)^2+4$

$$G(s) = \frac{2s+3}{(s+1)^2+4} = \frac{2(s+1)+1}{(s+1)^2+4}$$

$$= 2\underbrace{\left[\frac{s+1}{(s+1)^2+4}\right]}_{\mathcal{L}\{e^{-t}\cos(2t)\}} + \underbrace{\left[\frac{1}{(s+1)^2+4}\right]}_{\text{not quite } \mathcal{L}\{e^{-t}\sin(2t)\} = \frac{2}{(s+1)^2+4}}$$

$$= 2\left[\frac{s+1}{(s+1)^2+4}\right] + \frac{1}{2}\left[\frac{2}{(s+1)^2+4}\right]$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \boxed{2e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t)}$$

II. Partial Fractions:

Often our solution for $X(s)$ is in the form: $X(s) = \frac{P(s)}{Q(s)}$ called a rational function

where $P(s)$ and $Q(s)$ are both polynomials and degree $P(s)$ < degree of $Q(s)$

GOAL: Expand $X(s) = \frac{P(s)}{Q(s)}$ using Partial Fractions

Rules for Partial Fractions:

Rule 1: (Linear Factors)

$$\frac{P(s)}{(s-a)^n} = \frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \dots + \frac{A_n}{(s-a)^n}$$

$\mathcal{L}^{-1}$

$\frac{A_1}{s-a} \longmapsto e^{at}$

$\frac{A_n}{(s-a)^n} \longrightarrow e^{at}t^{n-1}$

$\frac{A_1 s+B_1}{(s-a)^2+b^2} \longmapsto c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt)$

(Quadratic Factors)

$(s-a)^n$ $\quad s-a \quad \ldots$

$$\frac{A_1 s + B_1}{(s-a)^2+b^2} \mapsto c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt)$$

Rule 2: (Quadratic Factors)

$$\frac{P(s)}{[(s-a)^2+b^2]^n} = \frac{A_1 s + B_1}{(s-a)^2+b^2} + \frac{A_2 s + B_2}{[(s-a)^2+b^2]^2} + \cdots + \frac{A_n s + B_n}{[(s-a)^2+b^2]^n}$$

Ex: Find the inverse L.T. of $G(s) = \frac{5}{s^4+9s^2}$

$$G(s) = \frac{5}{s^4+9s^2} = \frac{5}{\underbrace{s^2}_{\text{linear}}\underbrace{(s^2+9)}_{\text{quadratic}}}$$

Rule 1: $\frac{1}{s^2} \longrightarrow \frac{A}{s} + \frac{B}{s^2}$

Rule 2: $\frac{1}{s^2+9} \longrightarrow \frac{Cs+D}{s^2+9}$

Partial Fractions:

$$\frac{5}{s^2(s^2+9)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+9}$$

multiply by common denominator $s^2(s^2+\ )$

$$5 = As(s^2+9) + B(s^2+9) + (Cs+D)s^2$$

Evaluate s at key points

$\boxed{@\, s=0}$ $\quad s^2=0$

$$5 = 0 + B(0+9) + 0 \longrightarrow \boxed{B = \frac{5}{9}}$$

$\boxed{@\, s=3i}$ $\quad s^2+9 = -9+9 = 0$

$$5 = 0 + 0 + (3iC+D)(3i)^2$$

$$5 = (3iC+D)(-9) = -27iC - 9D$$

$$5 = (3iC+D)(-9) = -27iC - 9D$$

$$5 = -9D \qquad\qquad 0 = -27iC$$

$$\boxed{D = \frac{-5}{9}} \qquad\qquad \boxed{C = 0}$$

$\boxed{@\ s=1}$ $\quad s^2=1 \quad s^2+9=10$

$$5 = A(1)(10) + \left(\frac{5}{9}\right)(10) + \left(0s - \frac{5}{9}\right)(1)^2$$

$$\cancel{5} = 10A + \frac{50}{9} - \frac{5}{9} = 10A + \frac{45}{9} = 10A + \cancel{5}$$

$$0 = 10A \longrightarrow \boxed{A=0}$$

so

$$G(s) = \frac{5}{s^2(s^2+9)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+9}$$

$$= \frac{0}{s} + \frac{5}{9}\left(\frac{1}{s^2}\right) + \frac{0\cdot s - \frac{5}{9}}{s^2+9}$$

$$= \frac{5}{9}\left(\frac{1}{s^2}\right) - \frac{5}{9}\left(\frac{1}{s^2+9}\right)$$

almost $\mathcal{L}\{\sin(3t)\} = \frac{3}{s^2+9}$

$$= \frac{5}{9}\left(\frac{1}{s^2}\right) - \frac{5}{27}\left(\frac{3}{s^2+9}\right)$$

$$\text{so } g(t) = \frac{5}{9}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{5}{27}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$= \boxed{\frac{5}{9}t - \frac{5}{27}\sin(3t)}$$

Ex: Find the inverse L.T. of $G(s) = \dfrac{60}{(s^2+4)((s+3)^2+25)}$

Rule 2 : $\quad \dfrac{1}{\ } \longrightarrow \dfrac{As+B}{\ }$ $\quad$ cos(2t) or sin(2t)

Rule 2 : $\dfrac{1}{s^2+4} \longrightarrow \dfrac{As+B}{s^2+4} \longleftarrow$ cos(2t) or sin(2t)

Rule 2 : $\dfrac{1}{(s+3)^2+25} \longrightarrow \dfrac{Cs+D}{(s+3)^2+25} \longleftarrow$ $e^{-3t}\cos(5t)$ or $e^{-3t}\sin(5t)$

Partial Fractions:

$$\frac{60}{(s^2+4)((s+3)^2+25)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{(s+3)^2+25}$$

multiply by common denom.

$$60 = (As+B)\left[(s+3)^2+25\right] + (Cs+D)(s^2+4)$$

Evaluate at points in s

$\boxed{s=2i}$ $\quad s^2+4=0$

$$60 = (2iA+B)\left[(3+2i)^2+25\right] + 0$$

$$= (2iA+B)\left[9+12i-4+25\right]$$

$$= (2iA+B)(30+12i)$$

$$= 60iA + 30B - 24A + 12iB$$

$$60 = (30B-24A) + i(60A+12B)$$

$$60 = 30B - 24A$$

$$= 30(-5A) - 24A$$

$$60 = -174A$$

$$A = \frac{-60}{174} = \boxed{\frac{-10}{29} = A}$$

$$0 = 60A + 12B$$

$$0 = 5A + B \longrightarrow B = -5A$$

$$\boxed{B = \frac{50}{29}}$$

$\boxed{@\ s = -3+5i}$ $(s+3)^2 + 25 = 0$

$$60 = 0 + [C(-3+5i) + D][(-3+5i)^2 + 4]$$

$$= [(-3+5i)C + D](-12-30i)$$

$$= (-3+5i)(-12-30i)C + (-12-30i)D$$

$$= (186+30i)C + (-12-30i)D$$

$$60 = (186C - 12D) + i(30C - 30D)$$

$$60 = 186C - 12D$$

$$= 186D - 12D$$

$$= 174D$$

$$D = \frac{60}{174} = \boxed{\frac{10}{29} = D}$$

$$0 = 30C - 30D$$

$$C = D$$

$$\boxed{C = \frac{10}{29}} \quad \boxed{B = \frac{50}{29}} \quad \boxed{A = \frac{-10}{29}}$$

$$G(s) = \frac{As+B}{s^2+4} + \frac{Cs+D}{(s+3)^2+25}$$

$$= \frac{-10}{29}\left[\frac{s-5}{s^2+4}\right] + \frac{10}{29}\left[\frac{s+1}{(s+3)^2+25}\right]$$

$$= \frac{-10}{29}\left[\underbrace{\left(\frac{s}{s^2+4}\right)}_{\mathcal{L}\{\cos(2t)\}} - \frac{5}{2}\underbrace{\left(\frac{2}{s^2+4}\right)}_{\mathcal{L}\{\sin(2t)\}}\right]$$

$$+ \frac{10}{29}\left[\underbrace{\left(\frac{s+3}{(s+3)^2+25}\right)}_{\mathcal{L}\{e^{-3t}\cos(5t)\}} - \frac{2}{5}\underbrace{\left(\frac{5}{(s+3)^2+25}\right)}_{\mathcal{L}\{e^{-3t}\sin(5t)\}}\right]$$

$$g(t) = \frac{-10}{29}\left[\cos(2t) - \frac{5}{2}\sin(2t)\right] + \frac{10}{29}\left[e^{-3t}\cos(5t) - \frac{2}{5}e^{-3t}\sin(5t)\right]$$

$$\boxed{g(t) = \frac{5}{29}\left[-2\cos(2t) + 5\sin(2t)\right] + \frac{2}{29}e^{-3t}\left[5\cos(5t) - 2\sin(5t)\right]}$$

★ Summary:

- Translation on the s-axis

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

and conversely

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$$

- Partial Fractions:

Rule 1:
$$\frac{P(s)}{(s-a)^n} = \frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \dots + \frac{A_n}{(s-a)^n}$$

Rule 2:
$$\frac{P(s)}{[(s-a)^2+b^2]} = \frac{A_1 s + B_1}{(s-a)^2+b^2} + \dots + \frac{A_n s + B_n}{[(s-a)^2+b^2]^n}$$