☆ Derivatives, Integrals, and Products of Transforms

Warm up: Which of the following statements are true about the Laplace transform? (Check all that apply)

✓ (a) $\mathcal{L}\{a f(t)\} = a F(s)$

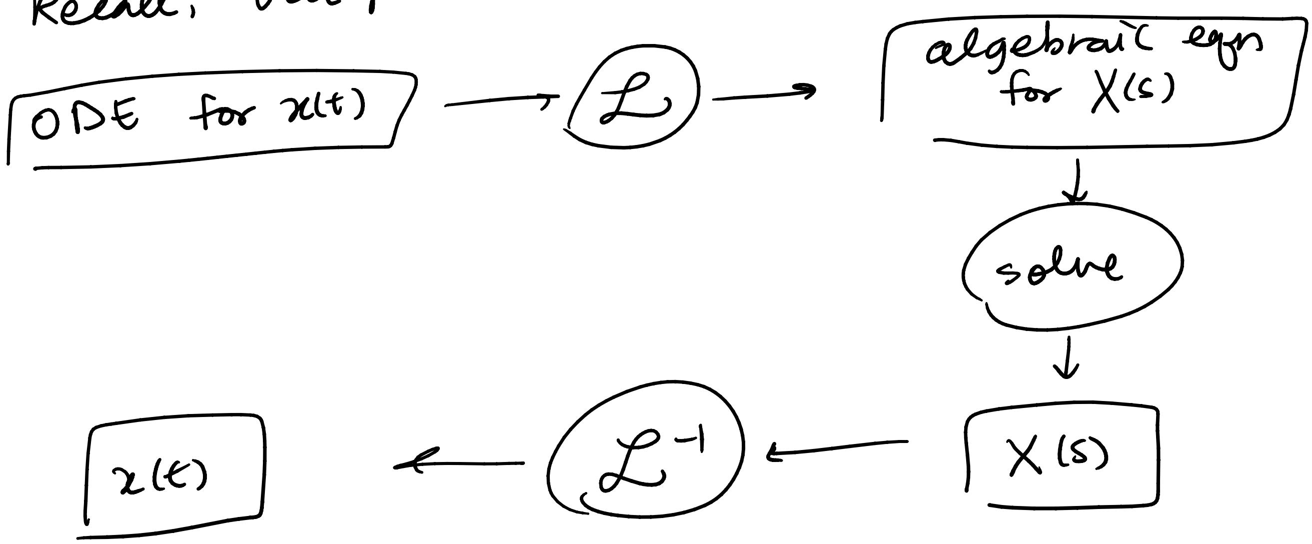
✗ (b) $\mathcal{L}\{[f(t)]^2\} = [F(s)]^2$

✓ (c) $\mathcal{L}\{f(t) + g(t)\} = F(s) + G(s)$

✗ (d) $\mathcal{L}\{f(t) g(t)\} = F(s) G(s)$

I. Convolutions:

Recall, our procedure for solving ODE using the L.T.



our solution $X(s)$ often has the form:

$$X(s) = F(s) G(s)$$

So often we want to find

$$x(t) = \mathcal{L}^{-1}\{F(s) G(s)\}$$

But, its NOT true that

$$\mathcal{L}^{-1}\{F(s) G(s)\} \neq f(t) g(t)$$

But, its NOT true that

$$\cancel{\mathcal{L}^{-1}\{F(s)\,G(s)\} = f(t)\,g(t)}$$

Q: What is $\mathcal{L}^{-1}\{F(s)G(s)\}$?

Def: The convolution of two functions $f(t)$ and $g(t)$ is:

$$(f*g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$
$$= \int_0^t f(t-\tau)\,g(\tau)\,d\tau$$

NOTE: order of fcns doesn't matter.

Thm (Convolution Property)

$$\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\} = F(s)G(s)$$

and conversely

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = (f*g)(t)$$

EX: Find the inverse L.T. $H(s) = \dfrac{2}{(s+1)(s-3)}$

NOTE: we could use partial fractions

$$\frac{2}{(s+1)(s-3)} = \frac{A}{s+1} + \frac{B}{s-3}$$

Instead, use convolution property

$$H(s) = \underbrace{\left(\frac{1}{s+1}\right)}_{F(s)}\underbrace{\left(\frac{2}{s-3}\right)}_{G(s)}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

$$g(t) = \mathcal{L}^{-1}\left\{\frac{2}{s-3}\right\} = 2e^{3t}$$

Thm

$$\mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t)$$

$$= \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$= \int_0^t e^{-\tau}\left(2e^{3(t-\tau)}\right) d\tau$$

$$= 2e^{3t} \int_0^t e^{-\tau-3\tau}\, d\tau = 2e^{3t}\int_0^t e^{-4\tau}\, d\tau$$

$$= 2e^{3t}\left[\frac{e^{-4\tau}}{-4}\right]_0^t = \frac{2}{-4}e^{3t}\left[e^{-4t} - 1\right]$$

$$= \frac{1}{2}e^{3t}\left[1 - e^{-4t}\right] = \boxed{\frac{1}{2}\left[e^{3t} - e^{-t}\right]}$$

Exercise: Double check using partial fractions

$$\frac{2}{(s+1)(s-3)} = \frac{A}{s+1} + \frac{B}{s-3} \longrightarrow \begin{array}{l} A = \frac{-1}{2} \\ B = \frac{1}{2} \end{array}$$

We can also the convolution property to write down solns to ODEs.

Ex: Apply the convolution property to write down the soln to:

$$x'' + 4x = f(t) \qquad x(0) = x'(0) = 0$$

1. Take L.T. of both sides

$$\mathcal{L}\{x''\} + 4\mathcal{L}\{x\} = \mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{x''\} + 4\mathcal{L}\{x\} = \mathcal{L}\{f(t)\}$$

$$[s^2X(s) - 0\cdot s - 0] + 4[X(s)] = F(s)$$

2. Solve for $X(s)$

$$(s^2+4)X(s) = F(s)$$

$$X(s) = \frac{F(s)}{s^2+4} = F(s)\left(\frac{1}{s^2+4}\right)$$

3. Take the inverse L.T.

let $G(s) = \frac{1}{s^2+4} \longrightarrow g(t) = \frac{1}{2}\sin(2t)$

$$x(t) = \mathcal{L}^{-1}\{F(s)G(s)\} \overset{\text{Thm}}{=} (f*g)(t)$$

$$= \int_0^t f(t-\tau)g(\tau)\,d\tau$$

$$\boxed{x(t) = \frac{1}{2}\int_0^t f(t-\tau)\sin(2\tau)\,d\tau}$$

II. Differentiation:

Recall: (Transforms of derivatives)

$$\mathcal{L}\{x'(t)\} = sX(s) - x(0)$$

t	s
derivative in t	multiplication by s

The reverse is "almost" true

t	s
multiply by $-t$	derivative in s

$$-te^{-st} = \frac{d}{ds}(e^{-st})$$

Thm (Differentiation)

$$\mathcal{L}\{-t f(t)\} = F'(s) = \frac{dF}{ds}$$

and conversely

and conversely

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}\{F'(s)\}$$

<u>Ex</u>: Find the L.T. of $g(t) = t\cosh(2t)$

here, let $f(t) = -\cosh(2t)$

so $F(s) = \mathcal{L}\{-\cosh(2t)\} = \dfrac{-s}{s^2-4}$

Then $\mathcal{L}\{g(t)\} = \mathcal{L}\{-t\,f(t)\} \overset{\text{Thm}}{=} \dfrac{dF}{ds}$

$$= \frac{d}{ds}\left(\frac{-s}{s^2-4}\right)$$

$$= (-1)\left(\frac{1}{s^2-4}\right) + (-s)(-1)(s^2-4)^{-2}(2s)$$

$$= \frac{-1}{s^2-4}\frac{(s^2-4)}{(s^2-4)} + \frac{2s^2}{(s^2-4)^2}$$

$$= \frac{-s^2+4+2s^2}{[s^2-4]^2} = \boxed{\frac{s^2+4}{[s^2-4]^2}}$$

III. Integration:

<u>Recall</u>: transforms of integrals:

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}$$

t	s
integrate in t	division by s

The reverse is true

t	s
division by t	integrate in s

Thm: (Integration of transforms)

NOTE: indefinite

Thm: (Integration of transforms)

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\,d\sigma$$

NOTE: indefinite integral

and conversely

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = t\,\mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma)\,d\sigma\right\}$$

Ex: Find $\mathcal{L}\left\{\frac{\sinh(t)}{t}\right\}$

$f(t) = \sinh(t)$ $\qquad F(s) = \mathcal{L}\{\sinh(t)\} = \frac{1}{s^2-1}$

then $\mathcal{L}\left\{\frac{\sinh(t)}{t}\right\} \overset{\text{Thm}}{=} \int_s^\infty F(\sigma)\,d\sigma = \int_s^\infty \frac{d\sigma}{\sigma^2-1}$

$= \int_s^\infty \frac{d\sigma}{(\sigma-1)(\sigma+1)}$ ← Expand this using Partial Fractions

$$\frac{1}{(\sigma-1)(\sigma+1)} = \frac{A}{\sigma-1} + \frac{B}{\sigma+1}$$

$$1 = A(\sigma+1) + B(\sigma-1)$$

$\boxed{@\,\sigma=1}$ $\quad \sigma-1=0 \quad \sigma+1=2$

$1 = 2A \longrightarrow A = 1/2$

$\boxed{@\,\sigma=-1}$ $\quad \sigma+1=0 \quad \sigma-1=-2$

$1 = 0 - 2B \longrightarrow B = -1/2$

$$\frac{1}{(\sigma-1)(\sigma+1)} = \frac{1/2}{\sigma-1} + \frac{-1/2}{\sigma+1}$$

$$= \frac{1}{2}\int_s^\infty \frac{d\sigma}{\sigma-1} - \frac{1}{2}\int_s^\infty \frac{d\sigma}{\sigma+1} = \lim_{b\to\infty}\frac{1}{2}\left[\int_s^b \frac{d\sigma}{\sigma-1} - \int_s^b \frac{d\sigma}{\sigma+1}\right]$$

$$= \frac{1}{2}\lim_{b\to\infty}\left[\left(\ln(\sigma-1)\right)\Big|_s^b - \left(\ln(\sigma+1)\right)\Big|_s^b\right]$$

$$= \frac{1}{2}\lim_{b\to\infty}\Big[\big(\ln(\sigma-1)\big)_s - \big(\ \dots\ \big)\big|_s \Big]$$

$$= \frac{1}{2}\lim_{b\to\infty}\Big[\ln(b-1) - \ln(s-1) - \ln(b+1) + \ln(s+1) \Big]$$

$$= \frac{1}{2}\lim_{b\to\infty}\Big[\underbrace{\ln\Big(\underbrace{\frac{b-1}{b+1}}_{\xrightarrow[\text{as } b\to\infty]{} 1}\Big)}_{\longrightarrow 0} + \ln\Big(\frac{s+1}{s-1}\Big) \Big] = \boxed{\frac{1}{2}\ln\Big(\frac{s+1}{s-1}\Big)}$$

$$\boxed{\mathcal{L}\Big\{\frac{\sinh(t)}{t}\Big\} = \frac{1}{2}\ln\Big|\frac{s+1}{s-1}\Big|}$$

NOTE: Sometimes its easier to use the Integral Property instead of partial fractions

Ex: Find $\mathcal{L}^{-1}\Big\{\frac{2s}{[s^2-1]^2}\Big\}$

Partial Fractions expansion

Rule 2: $\frac{2s}{[s^2-1]^2} = \frac{As+B}{s^2-1} + \frac{Cs+D}{[s^2-1]^2}$

Instead, we recognize that $F(s)$ can be integrated

$$\mathcal{L}^{-1}\Big\{\frac{2s}{[s^2-1]^2}\Big\} \overset{\text{Thm}}{=} t\,\mathcal{L}^{-1}\Big\{\int_s^\infty \frac{2\sigma\, d\sigma}{[\sigma^2-1]^2}\Big\}$$

$u = \sigma^2 - 1$

$du = 2\sigma\, d\sigma$

$$= t\,\mathcal{L}^{-1}\Big\{\int_{s^2-1}^\infty \frac{du}{u^2}\Big\}$$

$$= t\,\mathcal{L}^{-1}\Big\{\lim_{b\to\infty} \int^b \frac{du}{u^2}\Big\}$$

$$= t\mathcal{L}^{-1}\left\{ \lim_{b\to\infty} \int_{s^2-1}^{b} \frac{du}{u^2} \right\}$$

$$= t\mathcal{L}^{-1}\left\{ \lim_{b\to\infty} \left[\frac{u^{-1}}{-1} \right]_{s^2-1}^{b} \right\}$$

$$= t\mathcal{L}^{-1}\left\{ \lim_{b\to\infty} \left[\underbrace{-\frac{1}{b}}_{\to 0 \text{ as } b\to\infty} - \frac{-1}{s^2-1} \right] \right\}$$

$$= t\mathcal{L}^{-1}\left\{ \frac{1}{s^2-1} \right\} = \boxed{t\sinh(t) = \mathcal{L}\left\{ \frac{2s}{[s^2-1]^2} \right\}}$$

☆ Summary:

- Convolution Property

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t)$$

$$= \int_0^t f(t-\tau)g(\tau)d\tau$$

- Differentiation

$$\mathcal{L}\{-t f(t)\} = F'(s)$$

and conversely

$$\mathcal{L}^{-1}\{F(s)\} = \frac{-1}{t}\mathcal{L}\{F'(s)\}$$

- Integration

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)d\sigma$$

and conversely

$$\mathcal{L}^{-1}\{F(s)\} = t\mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma)d\sigma\right\}$$

$$\mathcal{L}^{-1}\{F(s)\} = t\,\mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma)\,d\sigma\right\}$$