Section 7.5

☆ Piecewise Continuous Input Functions

Warm up: Recall, the unit step function:

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

What is $\mathcal{L}\{u(t-3)\} = \dfrac{e^{-3s}}{s}$

We've been using Laplace Transforms to solve ODEs:

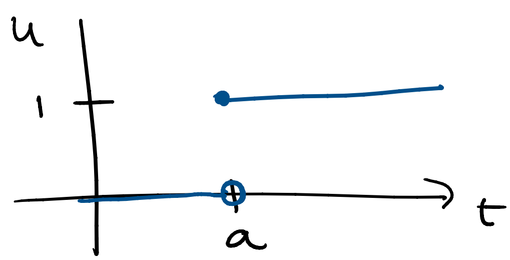
$$ax'' + bx' + cx = f(t)$$

One advantage of L.T. we can solve when $f(t)$ is piecewise continuous

I. Piecewise Continuous Functions:

unit step function

$$u(t-a) = u_a(t) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$



"flipping on a switch at $t=a$"

In Sec 7.1, we derived

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

We can think of this as:

$$\mathcal{L}\{u(t-a)\,(1)\} = e^{-as}\mathcal{L}\{1\} = \frac{e^{-as}}{s}$$

Thm (Translation on the t-axis)

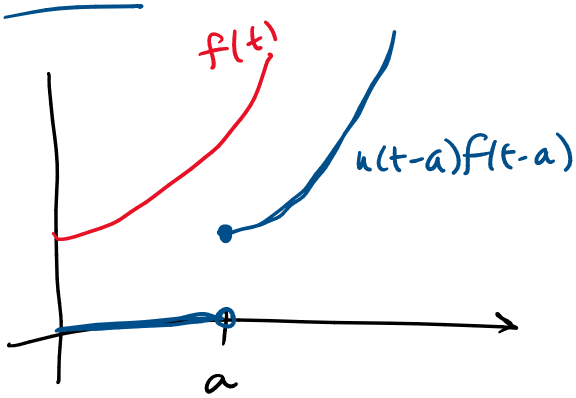
$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$$

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as} F(s)$$

and conversely

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a)$$

NOTE: $u(t-a)\,f(t-a) = f(t-a)\begin{cases} 0 & t<a \\ 1 & t \geq a \end{cases}$



$$= \begin{cases} 0 & t<a \\ f(t-a) & t \geq a \end{cases}$$

"time delay of a"

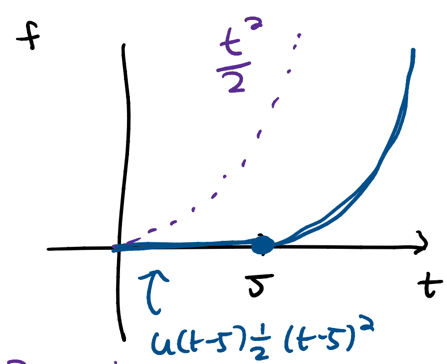
Ex: Find $\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s^3}\right\} = \mathcal{L}^{-1}\left\{e^{-5s}\underbrace{\left(\frac{1}{s^3}\right)}_{F(s)}\right\}$

Thm $\rightarrow$ $\stackrel{?}{=} u(t-5)\,f(t-5)$

$F(s) = \frac{1}{s^3}$ $\qquad f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = \frac{t^2}{2}$

$$= \boxed{u(t-5)\left[\frac{1}{2}(t-5)^2\right]}$$

$$= \begin{cases} 0 & t<5 \\ \frac{1}{2}(t-5)^2 & t \geq 5 \end{cases}$$



Ex: Find $\mathcal{L}\{g(t)\}$ where

$$g(t) = \begin{cases} 0 & t<3 \\ t^2 & t \geq 3 \end{cases}$$

$g(t) = t^2\,u(t-3)$

$g(t) = t^2 u(t-3)$

To apply the Thm, Need to write in form:

$$g(t) = f(t-3)u(t-3)$$

WANT: $f(t-3) = t^2$

let $t \longmapsto t+3$

$$f((t+3)-3) = (t+3)^2$$

$$f(t) = (t+3)^2$$

$$g(t) = f(t-3)u(t-3) = (t-3+3)^2 u(t-3) = t^2 u(t-3)$$

Apply the Thm:

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{f(t-3)u(t-3)\} \overset{\text{Thm}}{=} e^{-3s}F(s)$$

$$= e^{-3s}\mathcal{L}\{(t+3)^2\} = e^{-3s}\mathcal{L}\{t^2+6t+9\}$$

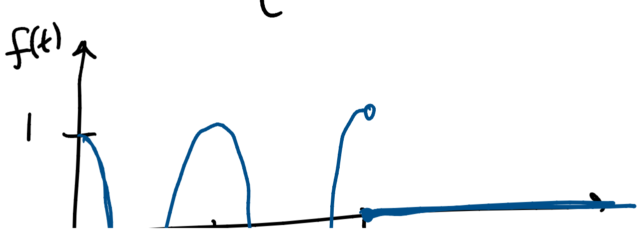
linearity $= e^{-3s}\left[\mathcal{L}\{t^2\} + 6\mathcal{L}\{t\} + 9\mathcal{L}\{1\}\right]$

$$= e^{-3s}\left[\frac{2!}{s^3} + 6\left(\frac{1}{s^2}\right) + 9\left(\frac{1}{s}\right)\right]$$

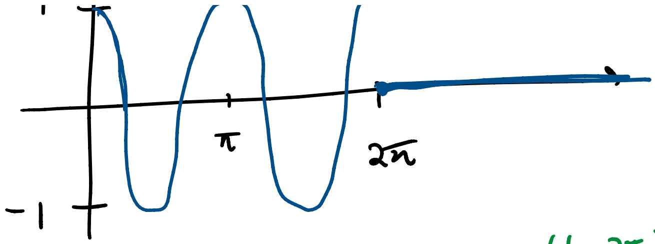
$$= \boxed{e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]}$$

Ex: Find the L.T. of

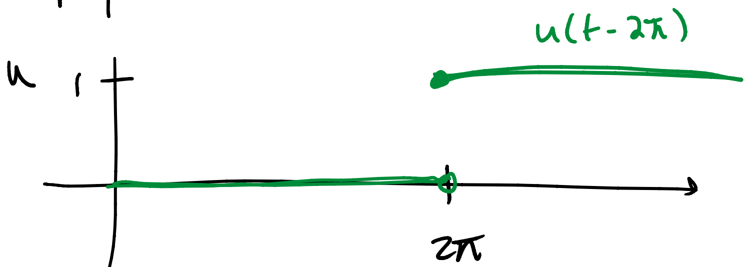
$$f(t) = \begin{cases} \cos(2t) & 0 < t < 2\pi \\ 0 & t \geq 2\pi \end{cases}$$



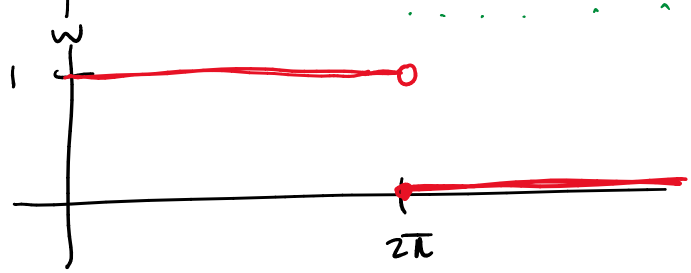
GOAL: write $f(t)$ interms of $u(t-2\pi)$



of $u(t-2\pi)$



want opposite of this $w(t-2\pi)$



$w(t-2\pi)$

$f(t) = \sin(2t)\, w(t-2\pi)$

Let $w(t-2\pi) = 1 - u(t-2\pi)$

$$= 1 - \begin{Bmatrix} 0 & t < 2\pi \\ 1 & t \geq 2\pi \end{Bmatrix}$$

$$= \begin{Bmatrix} 1-0 & t < 2\pi \\ 1-1 & t \geq 2\pi \end{Bmatrix}$$

$$= \begin{Bmatrix} 1 & t < 2\pi \\ 0 & t \geq 2\pi \end{Bmatrix} = w(t-2\pi)$$

So now, write

$$f(t) = \sin(2t)\, w(t-2\pi) = \sin(2t)[1 - u(t-2\pi)]$$

$$= \sin(2t) \begin{Bmatrix} 1 & t < 2\pi \\ 0 & t \geq 2\pi \end{Bmatrix} = \begin{Bmatrix} \sin(2t) & t < 2\pi \\ 0 & t \geq 2\pi \end{Bmatrix}$$

Lastly, we know that $\sin(2t)$ is periodic with the period π

$$\sin(2t) = \sin(2(t-\pi)) = \sin(2(t-2\pi))$$

with the period ..

$$\sin(2t) = \sin(2(t-\pi)) = \sin(2(t-2\pi))$$

$$f(t) = \sin(2t)\,[1 - u(t-2\pi)]$$

$$= \sin(2t) - \sin(2t)\,u(t-2\pi)$$

b/c $\sin(2t)$ is periodic

$$= \sin(2t) - \sin(2(t-2\pi))\,u(t-2\pi)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin(2t)\} - \mathcal{L}\{\sin(2(t-2\pi))\,u(t-2\pi)\}$$

Thm

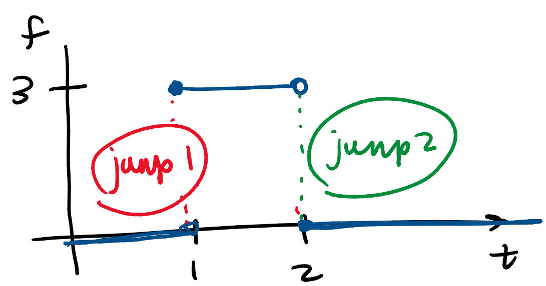
$$= \mathcal{L}\{\sin(2t)\} - e^{-2\pi s}\,\mathcal{L}\{\sin(2t)\}$$

$$= \left(\frac{2}{s^2+4}\right) - e^{-2\pi s}\left(\frac{2}{s^2+4}\right)$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{2(1-e^{-2\pi s})}{s^2+4}}$$

★ <u>Claim</u>: We can write <u>any</u> piecewise continuous function in terms of unit step functions.

<u>Ex</u>: $f(t) = \begin{cases} 0 & t < 1 \\ 3 & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}$

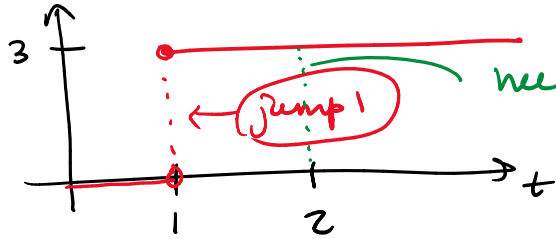


Two discontinuities (jumps) at $t=1$ and $t=2$

Know we need $u(t-1)$ and $u(t-2)$

jump 1 is represented by $3u(t-1) = 3\begin{cases} 0 & t<1 \\ 1 & t \ge 1 \end{cases}$

jump 1 is represented by $3u(t-1)$ … $t \ge 1$



need "switch off" at $t=2$.

To find jump 2

$$\underbrace{3u(t-1)}_{\text{jump 1}} - f(t) = \begin{cases} 0 & t<1 \\ 3 & 1 \le t < 2 \\ 3 & t \ge 2 \end{cases} - \begin{cases} 0 & t<1 \\ 3 & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}$$

jump 2

$$= \begin{cases} 0-0 & t<1 \\ 3-3 & 1 \le t < 2 \\ 3-0 & t \ge 2 \end{cases} = \begin{cases} 0 & t<1 \\ 0 & 1 \le t < 2 \\ 3 & t \ge 2 \end{cases}$$

$$= \begin{cases} 0 & t<2 \\ 3 & t \ge 2 \end{cases} = 3u(t-2)$$

$$3u(t-1) - f(t) = 3u(t-2)$$

Rearrange

$$3u(t-1) - 3u(t-2) = f(t)$$

Take L.T.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{3u(t-1) - 3u(t-2)\}$$

linearity $\rightarrow$ $= 3\mathcal{L}\{u(t-1)\} - 3\mathcal{L}\{u(t-2)\}$

$$= 3\frac{e^{-s}}{s} - \frac{3e^{-2s}}{s} = \boxed{\frac{3}{s}\left[e^{-s} - e^{-2s}\right]}$$

Ex. Find the inverse L.T. of

Ex: Find the inverse L.T. of

$$F(s) = \frac{s(1+e^{-3s})}{s^2+\pi^2}$$ and sketch $f(t)$

$$\mathcal{L}^{-1}\left\{\frac{s(1+e^{-3s})}{s^2+\pi^2}\right\} = \mathcal{L}^{-1}\left\{\underbrace{\frac{s}{s^2+\pi^2}}_{G(s)}\right\} + \mathcal{L}^{-1}\left\{e^{-3s}\underbrace{\left(\frac{s}{s^2+\pi^2}\right)}_{G(s)}\right\}$$

Thm $\rightarrow$ $= g(t) + u(t-3)g(t-3)$

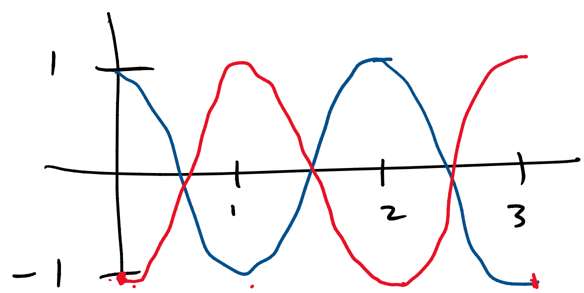
$$g(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+\pi^2}\right\} = \cos(\pi t)$$

$$= \cos(\pi t) + u(t-3)\cos(\pi(t-3))$$

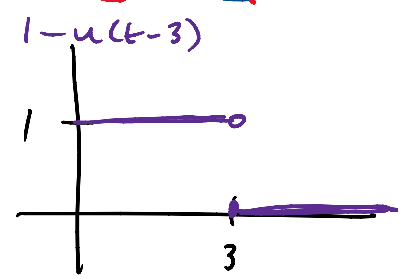
$$= \cos(\pi t) + u(t-3)\cos(\pi t - 3\pi)$$

period of $\cos(\pi t)$ is 2

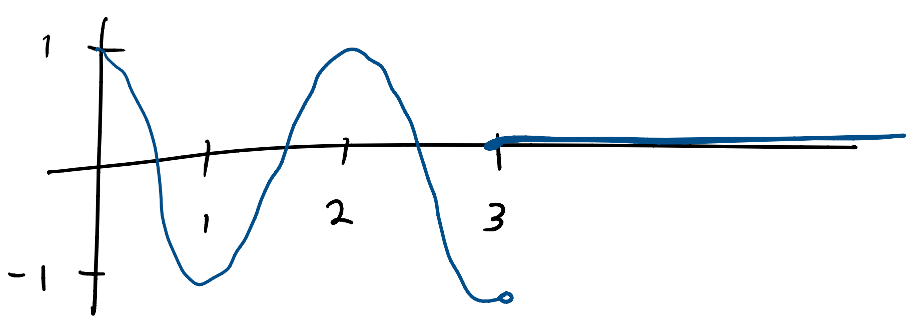
$\cos(\pi t - 3\pi) = -\cos(\pi t)$



$$= \cos(\pi t) - \cos(\pi t)\,u(t-3)$$



$$\boxed{f(t) = \cos(\pi t)\left[1 - u(t-3)\right]}$$



☆ **Summary:**

- **Thm:** (Translation on the t-axis)

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$$

and conversely

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a)$$

- $$[1 - u(t-a)] = \begin{cases} 1 & t < a \\ 0 & t \geq a \end{cases}$$

- Any piecewise continuous function can be represented in terms of unit step functions.