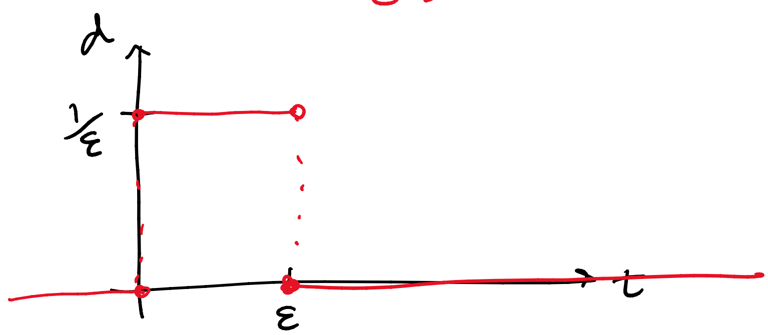
Section 7.6

# Impulses & Delta Functions

**Warm up:** Plot the piecewise continuous function

$$d_\varepsilon(t) = \frac{1}{\varepsilon}\left[u(t) - u(t-\varepsilon)\right]$$




## I. Delta Function:

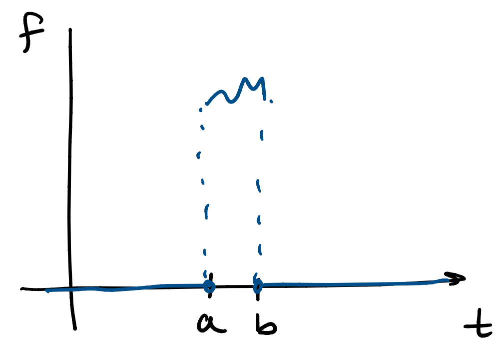
**Motivation:**

impulsive force

acts on a short time

e.g.: - bat striking a ball

- surge of voltage



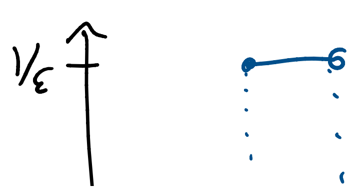

The effect depends on the impulse $p$ of the function $f(t)$ over the interval $[a,b]$

$$p = \int_a^b f(t)\,dt$$

Q: How do we model this?

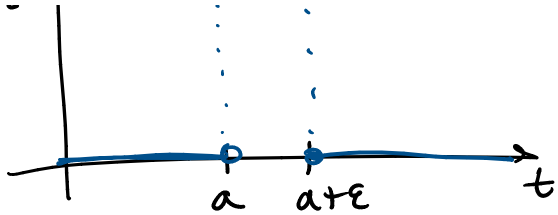
**Def:** The unit impulse $d_{a,\varepsilon}(t)$ is defined

$$d_{a,\varepsilon}(t) = \begin{cases} 1/\varepsilon & \text{if } a \le t < a+\varepsilon \\ 0 & \text{otherwise} \end{cases}$$




as $\varepsilon \to 0$

$$\frac{1}{\varepsilon} \longrightarrow +\infty$$




$\frac{1}{\varepsilon} \longrightarrow + \ldots$

impulse: $p = \int_0^\infty d_{a,\varepsilon}(t)\,dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\,dt = \left[\frac{t}{\varepsilon}\right]_a^{a+\varepsilon}$

$= \frac{a+\varepsilon}{\varepsilon} - \frac{a}{\varepsilon} = \frac{\varepsilon}{\varepsilon} = 1$

Q: What happens as $\varepsilon \to 0$?

$\lim_{\varepsilon \to 0} d_{a,\varepsilon}(t) = \lim_{\varepsilon \to 0} \begin{cases} 1/\varepsilon & a \le t < a+\varepsilon \\ 0 & \text{otherwise} \end{cases}$

$= \begin{cases} +\infty & t = a \\ 0 & \text{otherwise} \end{cases} = \delta(t-a)$

impulse $p = \int_0^\infty \delta(t-a)\,dt = \lim_{\varepsilon \to 0} \int_0^\infty d_{a,\varepsilon}(t)\,dt$

$= \lim_{\varepsilon \to 0} 1 = 1$

Def: The Dirac delta function is $\delta(t-a)$

$\delta(t-a) = \begin{cases} +\infty & \text{if } t = a \\ 0 & \text{if } t \neq a \end{cases}$ and $\int_0^\infty \delta(t-a)\,dt = 1$


NOTE: $\delta(t-a)$ is not really a function because of

NOTE: $\delta(t-a)$ is not really a function but it is a useful idea because of the following property

$$\int_0^\infty g(t)\,\delta(t-a)\,dt = g(a)$$ ← picks out the value of $g$ at $t=a$

$g(t)$ function $\longrightarrow$ $\boxed{\int_0^\infty g(t)\,\delta(t-a)\,dt}$ $\longrightarrow$ $g(a)$ number

Q: What is $\mathcal{L}\{\delta(t-a)\} = ?$

plug in def:

$$\mathcal{L}\{\delta(t-a)\} = \int_0^\infty e^{-st}\,\delta(t-a)\,dt = \left[e^{-st}\right]\Big|_{t=a}$$

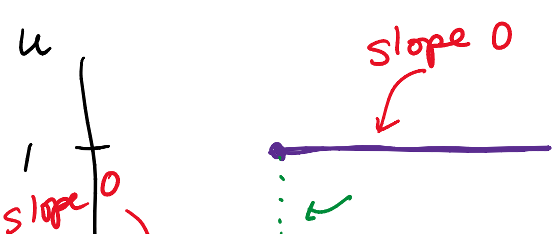
$$= e^{-as}$$

$$\boxed{\mathcal{L}\{\delta(t-a)\} = e^{-as}}$$

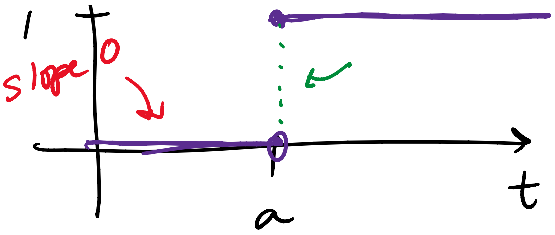
NOTE: We can also think of the Dirac delta as the derivative of the unit step function

$$\delta(t-a) = \begin{cases} +\infty & t=a \\ 0 & t\neq a \end{cases} = \frac{d}{dt}\left[u(t-a)\right]$$

$$= \frac{d}{dt}\left[\begin{cases} 0 & t<a \\ 1 & t\geq a \end{cases}\right]$$




$\{ \frac{d}{dt}(0)$ $t<a$




$$= \begin{cases} \frac{d}{dt}(0) & t<a \\ ? & t=a \\ \frac{d}{dt}(1) & t>a \end{cases}$$

$$\lim_{h\to 0} \frac{\delta(a+h)-\delta(a-h)}{2h}$$

$$\lim_{h\to 0} \frac{1-0}{2h} = +\infty$$

$$= \begin{cases} 0 & t<a \\ +\infty & t=a \\ 0 & t>a \end{cases}$$

$$= \begin{cases} +\infty & t=a \\ 0 & t\neq a \end{cases} = \delta(t-a)$$

use the fact $\delta(t-a) = \frac{d}{dt}\left[u(t-a)\right]$

$$\mathcal{L}\{\delta(t-a)\} = \mathcal{L}\left\{\frac{d}{dt}\left[u(t-a)\right]\right\}$$

$$= s\mathcal{L}\{u(t-a)\} - \cancel{u(0)} \to 0$$

$$= s\left[\frac{e^{-as}}{s}\right] = e^{-as} \checkmark$$

## II. IVPS:

GOAL: solve IVPs w/ forcing terms that are Dirac delta functions

Ex: $x'' + 4x = \delta(t-\pi)$ $\quad x(0) = x'(0) = 0$

@ $t=\pi$ there's a impulse

1. Take the L.T. of both sides

$$\mathcal{L}\{x''\} + 4\mathcal{L}\{x\} = \mathcal{L}\{\delta(t-\pi)\}$$

$-\pi s$

$$\left[s^2 X(s) - 0\cdot s - 0\right] + 4X(s) = e$$

2. Solve for $X(s)$

$$(s^2+4)\,X(s) = e^{-\pi s}$$

$$X(s) = \frac{e^{-\pi s}}{s^2+4}$$

exponentiel → a unit step fcns

3. Take the inverse L.T.

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{e^{-\pi s}\left(\frac{1}{s^2+4}\right)\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{e^{-\pi s}\underbrace{\left(\frac{2}{s^2+4}\right)}_{G(s)}\right\} \qquad g(t) = \sin(2t)$$

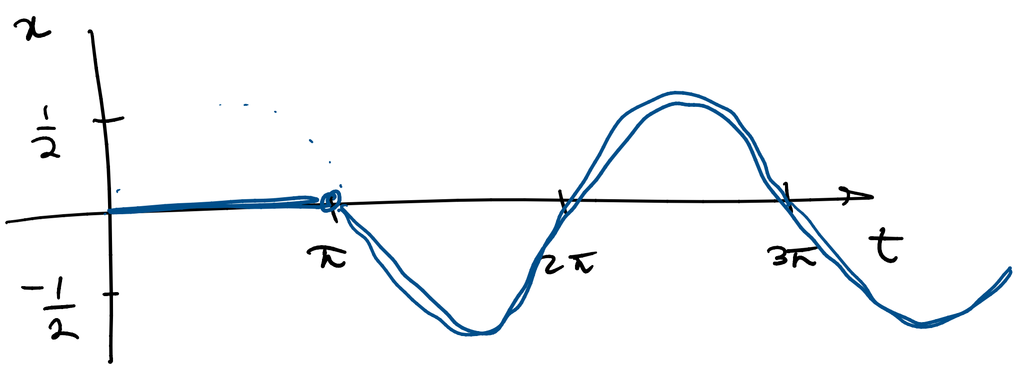
$$= \frac{1}{2}u(t-\pi)\,g(t-\pi)$$

$$= \frac{1}{2}\sin(2(t-\pi))\,u(t-\pi)$$

because $\sin(2t)$ is periodic →

$$= \frac{1}{2}\sin(2t-2\pi)\,u(t-\pi)$$

$$x(t) = \boxed{\frac{1}{2}\sin(2t)\,u(t-\pi)}$$




III. Duhamel's Principle:

ODE:

## III. Duhamel's Prinicple:

Consider a physical system by ODE:

$$a x'' + b x' + c x = f(t) \qquad x(0) = x'(0) = 0$$

$x(t)$ — output or response

$f(t)$ - input

1. Take the L.T. of both sides

$$a\left[s^2 X(s) - 0 \cdot s - 0\right] + b\left[s X(s) - 0\right] + c X(s) = F(s)$$

$$(as^2 + bs + c) X(s) = F(s)$$

2. Solve for $X(s)$

$$X(s) = \frac{F(s)}{as^2+bs+c} = F(s) \underbrace{\left(\frac{1}{as^2+bs+c}\right)}_{W(s)}$$

$$X(s) = F(s) W(s)$$

$$W(s) = \frac{1}{as^2+bs+c}$$ <u>transfer function</u> of the system

$$w(t) = \mathcal{L}^{-1}\{W(s)\}$$ <u>weight function</u>

3. Take the inverse L.T.

→ use the convolution property

$$x(t) = \mathcal{L}^{-1}\{W(s)F(s)\} \overset{\text{convolution property}}{=} (w * f)(t)$$

$$\boxed{x(t) = \int_0^t w(\tau) f(t-\tau)\, d\tau}$$

$$\boxed{x(t) = \int_0^t w(\tau) f(t-\tau)\, d\tau}$$

This is called Duhamel's Principle

Key step to solving the IVP: is finding

$$w(t) = \mathcal{L}^{-1}\{W(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{as^2+bs+c}\right\}$$

Ex: Apply Duhamel's Principle to write an integral formula for solution to the IVP

$$x'' + 6x' + 10 = f(t) \qquad x(0) = x'(0) = 0$$

1. Take the L.T.

$$s^2 X + 6sX + 10X = F(s)$$

2. Solve for $X(s)$

$$(s^2+6s+10)X(s) = F(s)$$

$$X(s) = F(s)\left(\frac{1}{s^2+6s+10}\right)$$

here $W(s) = \dfrac{1}{s^2+6s+10} = \dfrac{1}{(s+3)^2+1}$

3. Take the inverse L.T.

$$w(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+1}\right\} \leftarrow$$ Translation on s-axis Thm

$$= e^{-3t} \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$w(t) = e^{-3t}\sin(t)$$

So, by Duhamel's Priniuple

$$x(t) = \mathcal{L}^{-1}\{F(s)W(s)\} = (w*f)(t)$$

$$\boxed{x(t) = \int_0^t e^{-3\tau}\sin(\tau) f(t-\tau)\,d\tau}$$

IV. Final Exam Archives

Fall 2008 : #13

Find the solution of the IVP

$$y'' - 3y' + 2y = \delta(t-2), \quad y(0)=0, \quad y'(0)=1$$

1. Take the L.T. of both sides

$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-2)\}$$

$$[s^2Y - sy(0) - y'(0)] - 3[sY - y(0)] + 2Y = e^{-2s}$$

$$[s^2Y - 1] - 3(sY) + 2Y = e^{-2s}$$

$$(s^2 - 3s + 2)Y(s) - 1 = e^{-2s}$$

2. Solve for $Y(s)$

$$(s^2 - 3s + 2)Y(s) = 1 + e^{-2s}$$

$$Y(s) = \frac{1 + e^{-2s}}{\ldots}$$

$$Y(s) = \frac{1 + e^{-2s}}{s^2 - 3s + 2}$$

3. Take $\mathcal{L}^{-1}\{\cdot\}$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{1 + e^{-2s}}{(s-2)(s-1)} \right\}$$ ← factor denom.

linearity
$$= \mathcal{L}^{-1}\left\{ \underbrace{\frac{1}{(s-2)(s-1)}}_{G(s)} \right\} + \mathcal{L}^{-1}\left\{ e^{-2s} \left( \underbrace{\frac{1}{(s-2)(s-1)}}_{G(s)} \right) \right\}$$

$$G(s) = \frac{1}{(s-2)(s-1)} \qquad g(t) = \mathcal{L}^{-1}\{G(s)\}$$

then $y(t) = g(t) + u(t-2)\, g(t-2)$

could use partial fractions

$$g(t) = \mathcal{L}^{-1}\left\{ \frac{1}{(s-2)(s-1)} \right\}$$

use convolution property

$$\frac{1}{(s-2)(s-1)} = H(s)K(s)$$

$$H(s) = \frac{1}{s-2} \quad h(t) = e^{2t}$$

$$K(s) = \frac{1}{s-1} \quad k(t) = e^{t}$$

$$g(t) = \mathcal{L}^{-1}\{H(s)K(s)\} = (h * k)(t)$$ ← convolution property

$$= \int_0^t e^{2\tau} e^{(t-\tau)}\, d\tau = e^t \int_0^t e^{2\tau - \tau}\, d\tau$$

$$= e^t \int_0^t e^{\tau}\, d\tau = e^t \left[ e^{\tau} \right]_0^t = e^t \left[ e^t - 1 \right]$$

$$g(t) = e^{2t} - e^t$$

$$g(t) = e^{2t} - e^{t}$$

$$y(t) = g(t) + u(t-2)\,g(t-2)$$

$$= e^{2t} - e^{t} + u(t-2)\left[e^{2(t-2)} - e^{t-2}\right]$$

$$\boxed{y(t) = e^{2t} - e^{t} + u(t-2)\left[e^{2t-4} - e^{t-2}\right]}$$

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