Lecture 10 (Sec 7.2)

☆ Transformation of Initial Value Problems

Last lecture we defined the Laplace transform of a function $f(t)$ as:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

and the inverse Laplace transform:

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

And we wrote down a table of common L.T.s

Today, we will use the Laplace transform (LT) to solve initial value problems (IVPs)

Consider the const. coeff linear ODE:

$$ax'' + bx' + cx = f(t)$$

$$x(0) = x_0 \qquad x'(0) = x_0'$$

1. Start by taking the L.T. of both sides

$$\mathcal{L}\{ax'' + bx' + cx\} = \mathcal{L}\{f(t)\}$$

2. Since the L.T. is linear, we can write

$$a\mathcal{L}\{x''\} + b\mathcal{L}\{x'\} + c\mathcal{L}\{x\} = \mathcal{L}\{f(t)\}$$

<u>Q</u>: What is the L.T. of the derivative of a function $x'(t)$?

$$\mathcal{L}\{x'(t)\} = \int_0^\infty e^{-st} x'(t)\,dt$$

integrate by parts

$du = x'(t)dt \qquad v = e^{-st}$

$u = x(t) \qquad dv = -se^{-st}dt$

$$= \lim_{b\to\infty}\left[\left(x(t)e^{-st}\right)\Big|_0^b - \int_0^b -se^{-st}x(t)\,dt \right]$$

$$= \lim_{b\to\infty}\left[\underbrace{\left(x(b)e^{-sb}\right.}_{\text{converges to 0 if } s>0} - x(0)\Big] + \underbrace{s\int_0^b e^{-st}x(t)\,dt}_{\text{this is the definition of } \mathcal{L}\{x(t)\}} \right]$$

$$= -x(0) + s\,\mathcal{L}\{x(t)\}$$

$$\boxed{\mathcal{L}\{x'(t)\} = sX(s) - x(0) \qquad \text{if } s>0}$$

<u>Note</u>: we will use lower case letters for functions of t $\quad x(t),\ f(t), \ldots$
and use upper case letters for the Laplace transforms: $\quad X(s),\ F(s), \ldots$

We can repeat this procedure for higher order derivatives

$$\mathcal{L}\{x''(t)\} = s\,\mathcal{L}\{x'(t)\} - x'(0)$$

$$= s\left[sX(s) - x(0)\right] - x'(0)$$

$$= s^2X(s) - sx(0) - x'(0)$$

$$\mathcal{L}\{x'''(t)\} = s^3X(s) - s^2x(0) - sx'(0) - x''(0)$$

and so on

<u>Ex</u>: $x'' - x' - 6x = 0 \qquad x(0) = 2 \quad x'(0) = -1$

1. Take L.T. of both sides

$$\mathcal{L}\{x''\} - \mathcal{L}\{x'\} - 6\mathcal{L}\{x\} = 0$$

$$\left(s^2X(s) - sx(0) - x'(0)\right) - \left(sX(s) - x(0)\right) - 6X(s) = 0$$

$$s^2X - 2s - (-1) - sX + 2 - 6X = 0$$

2. Solve for $X(s)$

$$(s^2 - s - 6)X - 2s + 1 + 2 = 0$$

$$(s^2 - s - 6)X = 2s - 3$$

$$X(s) = \frac{2s-3}{s^2-s-6} = \underbrace{\frac{2s-3}{(s-3)(s+2)}}$$

we want to take the inverse L.T. of this term, but first we need to rewrite it in a more convenient form

3. Use the method of partial fractions

$$X(s) = \frac{2s-3}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$$

$$X(s) = \frac{2s-3}{(s-3)(s+2)} = \underbrace{\frac{A}{s-3}}_{} + \underbrace{\frac{B}{s+2}}_{}$$

this looks like $\mathcal{L}\{e^{3t}\} = \frac{1}{s-3}$

looks like $\mathcal{L}\{e^{-2t}\} = \frac{1}{s+2}$

Need to find A and B.

Multiply both sides by the common denominator $(s-3)(s+2)$

$$2s-3 = A(s+2) + B(s-3)$$

$$2s-3 = (A+B)s + (2A-3B)$$

$$2 = A+B$$

$$2-B = A$$

$$A = \frac{10}{5} - \frac{7}{5} = \frac{3}{5}$$

$$2A-3B = -3$$

$$2(2-B)-3B = -3$$

$$4-5B = -3$$

$$-5B = -7$$

$$B = \frac{7}{5}$$

so

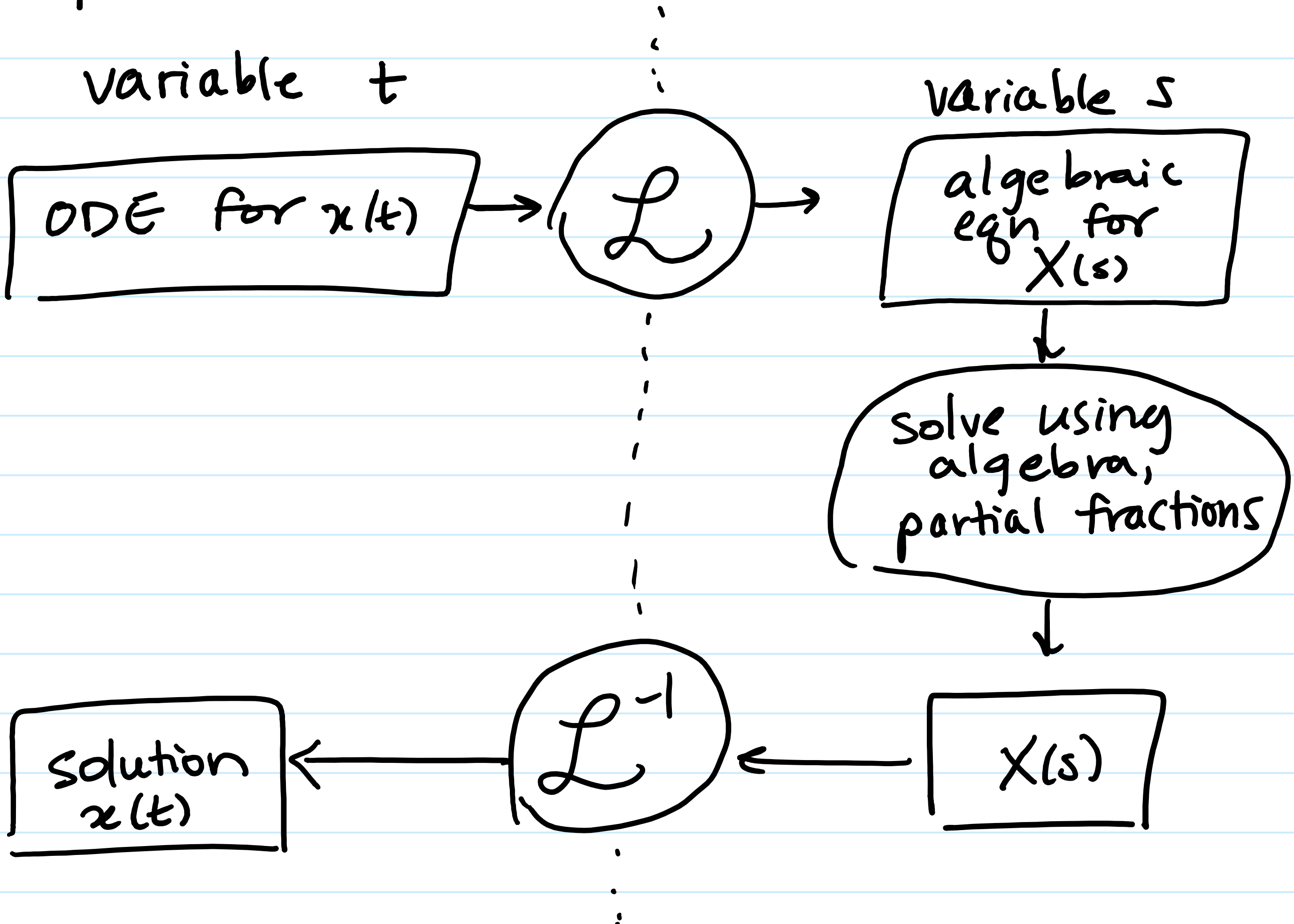
$$X(s) = \frac{3}{5}\left(\frac{1}{s-3}\right) + \frac{7}{5}\left(\frac{1}{s+2}\right)$$

4. Now take the inverse L.T.

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{3}{5}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) + \frac{7}{5}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)$$

$$\boxed{x(t) = \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}}$$

Laplace Transform Procedure:



the Laplace transform and inverse L.T. help us move back and forth between t and s variable

The L.T. <u>transforms</u> the ODE in t into an algebraic equation in s.

Ex: $x'' + x = \cos(2t)$ $x(0) = 0$, $x'(0) = 1$

1. Take the L.T. of both sides

$$\mathcal{L}\{x''\} + \mathcal{L}\{x\} = \mathcal{L}\{\cos(2t)\}$$

$$s^2 X - s\cancel{x(0)}^{0} - x'(0) + X = \frac{s}{s^2+4}$$

$$s^2 X - 1 + X = \frac{s}{s^2+4}$$

2. Solve for $X(s)$

$$(s^2+1)X - 1 = \frac{s}{s^2+4}$$

$$(s^2+1)X = \frac{s}{s^2+4} + 1$$

$$X(s) = \underbrace{\frac{s}{(s^2+4)(s^2+1)}}_{\text{rewrite using partial fractions}} + \frac{1}{s^2+1}$$

this is the L.T. of $\sin(t)$

3. Partial fractions

$$\frac{s}{(s^2+4)(s^2+1)} = \frac{A+Bs}{(s^2+4)} + \frac{C+Ds}{(s^2+1)}$$

$$\frac{s}{(s^2+4)(s^2+1)} = \frac{A+Bs}{s^2+4} + \frac{C+Ds}{s^2+1}$$

$$s = (A+Bs)(s^2+1) + (C+Ds)(s^2+4)$$

$$s = As^2 + A + Bs^3 + Bs + Cs^2 + 4C + Ds^3 + 4Ds$$

$$s = (B+D)s^3 + (A+C)s^2 + (B+4D)s + (A+4C)$$

$C+D=0$ | $A+C=0$ | $B+4D=1$ | $A+4C=0$
$B=-D$ | $A=-C$ | $-D+4D=1$ | $-C+4C=0$
$B=-\frac{1}{3}$ | $A=0$ | $3D=1$ | $C=0$
 | | $D=\frac{1}{3}$ |

$$\frac{s}{(s^2+4)(s^2+1)} = -\frac{1}{3}\left(\frac{s}{s^2+4}\right) + \frac{1}{3}\left(\frac{s}{s^2+1}\right)$$

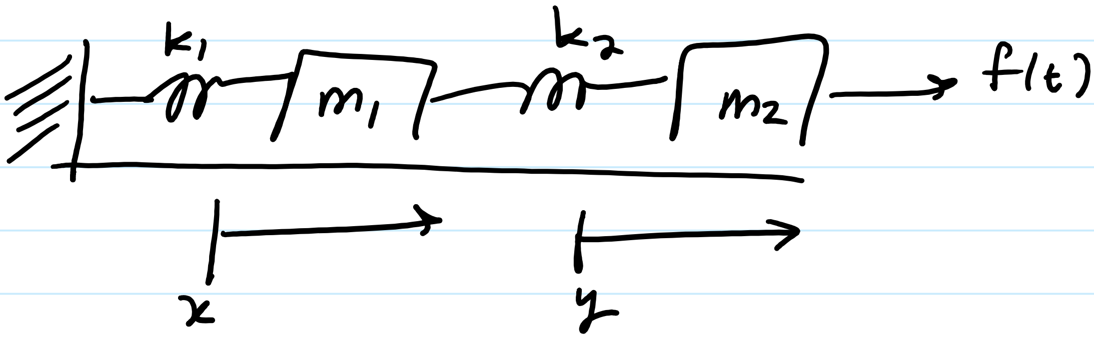
so $X(s) = -\frac{1}{3}\left(\frac{s}{s^2+4}\right) + \frac{1}{3}\left(\frac{s}{s^2+1}\right) + \frac{1}{s^2+1}$

4. Take the inverse L.T.

$$x(t) = -\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\boxed{x(t) = -\frac{1}{3}\cos(2t) + \frac{1}{3}\cos(t) + \sin(t)}$$

We can also use the Laplace transform to solve linear systems of ODE:



$$2x'' = -6x + 2y$$
$$y'' = 2x - 2y + 40\sin(3t)$$

with $x(0) = x'(0) = y(0) = y'(0) = 0$

1. Take L.T. of both sides

$X(s) = \mathcal{L}\{x\}$ $\qquad$ $Y(s) = \mathcal{L}\{y\}$

$$\mathcal{L}\{x''\} = s^2X - s\cancel{x(0)} - \cancel{x'(0)} = s^2X$$
$$\mathcal{L}\{y''\} = s^2Y - s\cancel{y(0)} - \cancel{y'(0)} = s^2Y$$
$$\mathcal{L}\{\sin(3t)\} = \frac{3}{s^2+9}$$

so we have

$$\begin{cases} 2s^2X = -6X + 2Y \\ s^2Y = 2X - 2Y + \dfrac{120}{s^2+9} \end{cases}$$

2. Collect like terms and solve for $X(s)$ and $Y(s)$

$$(2s^2+6)X - 2Y = 0$$
$$(s^2+3)X - Y = 0$$

$$(s^2+2)Y - 2X = \frac{120}{s^2+9}$$

so we have the algebraic system of eqns

$$(s^2+3)X \quad - Y \quad = 0$$
$$-2X \quad + (s^2+2)Y = \frac{120}{s^2+9}$$

1st eqn: $\quad Y = (s^2+3)X$

plug into 2nd eqn: $\quad -2X + (s^2+2)(s^2+3)X = \dfrac{120}{s^2+9}$

$$(s^4+3s^2+2s^2+6-2)X = \frac{120}{s^2+9}$$

$$(s^4+5s^2+4)X = \frac{120}{s^2+9}$$

$$(s^2+4)(s^2+1)X = \frac{120}{s^2+9}$$

$$X(s) = \frac{120}{(s^2+4)(s^2+1)(s^2+9)}$$

$$Y(s) = \frac{120(s^2+3)}{(s^2+4)(s^2+1)(s^2+9)}$$

3. Use partial fractions to expand

$$\frac{120}{(s^2+9)(s^2+4)(s^2+1)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+4} + \frac{Es+F}{s^2+1}$$

Since 120 on the left is a constant, we know that $A = C = E = 0$

$$\frac{120}{(s^2+9)(s^2+4)(s^2+1)} = \frac{B}{s^2+9} + \frac{D}{s^2+4} + \frac{F}{s^2+1}$$

Multiply by the common denominator

$$120 = B(s^2+4)(s^2+1) + D(s^2+9)(s^2+1) + F(s^2+9)(s^2+4)$$

plug in $s^2 = -9$

$$120 = B(-5)(-8) + D\cancel{(0)}^{\to 0}(-8) + F\cancel{(0)}^{\to 0}(-5)$$

$$3 = \frac{120}{40} = B$$

plug in $s^2 = -4$

$$120 = B\cancel{(0)}^{\to 0}(-3) + D(5)(-3) + F\cancel{(5)}^{\to 0}(0)$$

$$120 = D(-15) \rightarrow D = -8$$

plug in $s^2 = -1$

$$120 = B\cancel{(3)}^{\to 0}(0) + D\cancel{(8)}^{\to 0}(0) + F(8)(3)$$

$F = 120/24 = 5$

so $X(s) = \dfrac{3}{s^2+9} - \dfrac{8}{s^2+4} + \dfrac{5}{s^2+1}$

a similar process gives

$$Y(s) = \frac{-18}{s^2+9} + \frac{8}{s^2+4} + \frac{10}{s^2+1}$$

4. use inverse L.T. to get soln:

$$x(t) = \mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} - 8\,\mathcal{L}^{-1}\underbrace{\left\{\frac{1}{s^2+4}\right\}} + 5\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$\mathcal{L}\{\sin(2t)\} = \dfrac{2}{s^2+4}$

so need to pull in a factor of 2

$$= \mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} - 4\,\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} + 5\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\boxed{x(t) = 3\sin(3t) - 4\sin(2t) + 5\sin(t)}$$

$$y(t) = -6\,\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} + 4\,\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} + 10\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\boxed{y(t) = -6\sin(3t) + 4\sin(2t) + 10\sin(t)}$$

Thm: (Transform of Integrals)

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\left\{f(t)\right\} = \frac{F(s)}{s}$$

and conversely

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau$$

We can use this theorem to help us find inverse L.T.

Ex: Find the inverse L.T. of $G(s) = \dfrac{1}{s(s-3)}$

Want to use $\mathcal{L}^{-1}\left\{\dfrac{F(s)}{s}\right\} = \displaystyle\int_0^t f(\tau)\,d\tau$

Need to find $F(s)$ and $f(t)$

$$G(s) = \frac{1}{s(s-3)} = \frac{F(s)}{s} \Rightarrow F(s) = \frac{1}{s-3}$$

then $f(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s-3}\right\} = e^{3t}$

so

$$\mathcal{L}^{-1}\left\{G(s)\right\} = \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\}$$

$$= \int_0^t e^{3\tau}\,d\tau$$

$$= \left[\frac{e^{-3\tau}}{3}\right]_0^t = \frac{1}{3}\left[e^{-3t} - 1\right]$$

so $\mathcal{L}^{-1}\{G(s)\} = \frac{1}{3}\left[e^{3t} - 1\right]$

This method is more convenient than partial fractions when you want to find the inverse L.T. of a fcn like:

$$\frac{P(s)}{s^n Q(s)}$$