Lecture 11 (Sec 7.3)

☆ Translation & Partial Fractions

I. Translation

Thm: (Translation on the s-Axis)

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

and converseley

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$$

Proof:

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt = \int_0^\infty e^{-st}\left[e^{at} f(t)\right]dt = \mathcal{L}\{e^{at} f(t)\}$$

Ex: Find the L.T. of $g(t) = e^{3t}\cos(\pi t)$

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{e^{3t} f(t)\} = F(s-3) = \boxed{\frac{s-3}{(s-3)^2+\pi^2}}$$

$$f(t) = \cos(\pi t) \qquad F(s) = \frac{s}{s^2+\pi^2}$$

Table of Laplace Transforms:

$f(t)$	$F(s)$	
$e^{at} t^n$	$\frac{n!}{(s-a)^{n+1}}$	$s>a$
$e^{at}\cos(kt)$	$\frac{s-a}{(s-a)^2+k^2}$	$s>a$
$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$	$s>a$

Ex: Find the inverse L.T. $\quad G(s) = \frac{2s+3}{s^2+2s+5}$

Notice $s^2+2s+5 = (s+1)^2 + 4$

$$G(s) = \frac{2s+3}{(s+1)^2+4} = \frac{2(s+1)+1}{(s+1)^2+4}$$

$$= 2\underbrace{\left[\frac{s+1}{(s+1)^2+4}\right]}_{\mathcal{L}\{e^{-t}\cos(2t)\}} + \underbrace{\left[\frac{1}{(s+1)^2+4}\right]}_{\text{almost } \mathcal{L}\{e^{-t}\sin(2t)\}}$$

$$= 2\left[\frac{s+1}{(s+1)^2+4}\right] + \frac{1}{2}\left[\frac{2}{(s+1)^2+4}\right]$$

$$\boxed{g(t) = 2e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t)}$$

II. Rules for Partial Fractions

When solving

$$ax'' + bx' + cx = f(t) \qquad x(0)=x_0 \quad x'(0)=x'_0$$

1. Take L.T. of both sides
2. Solve for $X(s) = \dfrac{P(s)}{Q(s)}$ — rational function, degree $P(s)$ < degree $Q(s)$
3. Partial fractions to simply
4. $x(t) = \mathcal{L}^{-1}\{X(s)\}$

Rules:

Rule 1: (Linear Factors)

$$\frac{P(s)}{(s-a)^n} = \frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \dots + \frac{A_n}{(s-a)^n}$$

Rule 2: (Quadratic Factors)

$$\dots + \dots + \dots + \frac{A_n s + B_n}{\dots}$$

Rule 2: (Quadratic Factors)

$$\frac{P(s)}{[(s-a)^2+b^2]^n} = \frac{A_1 s + B_1}{(s-a)^2+b^2} + \frac{A_2 s + B_2}{[(s-a)^2+b^2]^2} + \dots + \frac{A_n s + B_n}{[(s-a)^2+b^2]^n}$$

Ex: Find the inverse L.T. of $G(s) = \frac{5}{s^4 + 9s^2}$

$$G(s) = \frac{5}{s^4+9s^2} = \frac{5}{\underbrace{s^2}_{\text{linear}}\underbrace{(s^2+9)}_{\text{quadratic}}}$$

use Rule 1

$$\frac{5}{s^2(s^2+9)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+9}$$

multiply by the common denom.

$$5 = As(s^2+9) + B(s^2+9) + (Cs+D)s^2$$

when $s=0 \rightarrow s^2=0 \quad s=0$

$$5 = 0 + B(9) + 0$$

$$B = 5/9$$

when $s=3i \rightarrow s^2+9 = -9+9=0$

$$5 = 0 + 0 + (C(3i)+D)(3i)^2$$

$$= (3iC + D)(-9)$$

$$5 = -27iC + (-9)D = -9D + i(-27C)$$

$$5 = -9D \qquad\qquad 0 = -27C$$

$$D = -5/9 \qquad\qquad C = 0$$

when $s=1 \rightarrow s^2+9 = 10$

$$5 = A(1)(10) + \frac{5}{9}\cdot 10 + (0 - 5/9)(1)^2$$

$$= 10A + \frac{50}{9} - \frac{5}{9} = 10A + \frac{45}{9} = 10A + 5$$

mistake

$$\cancel{0} = \cancel{10} = 10A$$

$$A = \cancel{x} \qquad \boxed{A=0}$$

$$G(s) = \cancel{\frac{1}{s}} + \frac{5}{9}\left[\frac{1}{s^2}\right] + \frac{\left[0s - \frac{5}{9}\right]}{s^2+9}$$

$$= \cancel{\frac{1}{s}} + \frac{5}{9}\left[\frac{1}{s^2}\right] - \frac{5}{9}\left[\frac{1}{s^2+9}\right]$$

$$= \cancel{\frac{1}{s}} + \frac{5}{9}\left[\frac{1}{s^2}\right] - \frac{5}{27}\left[\frac{3}{s^2+9}\right]$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \boxed{\cancel{x} + \boxed{\frac{5}{9}t - \frac{5}{27}\sin(3t)}}$$

Ex: Find the inverse L.T. of

$$G(s) = \frac{60}{(s^2+4)((s+3)^2+25)}$$

Using Rule 2 we expand

$$\frac{60}{(s^2+4)((s+3)^2+25)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{(s+3)^2+25}$$

multiply by the common denom.

$$60 = (As+B)((s+3)^2+25) + (Cs+D)(s^2+4)$$

when $s = 2i$ $\qquad s^2+4=0$

$$60 = (A(2i)+B)((2i+3)^2+25) + 0$$

$$= (2iA+B)[-4+6i+6i+9+25]$$

$$= (2iA+B)[30+12i]$$

$$= 60iA - 24A + 30B + 12iB$$

$$60 = (-24A+30B) + i(60A+12B)$$

$$60 = -24A + 30B \qquad\qquad 0 = 60A + 12B$$

$$60 = -24A + 30B$$
$$60 = -24A + 30(-5A)$$
$$= -24A - 150A$$
$$= -174A$$
$$A = \frac{-60}{174} = \frac{-10}{29}$$

$$0 = 60A + 12B$$
$$0 = 10A + 2B$$
$$0 = 5A + B$$
$$B = -5A$$
$$B = \frac{50}{29}$$

$$60 = (As+B)((s+3)^2+25) + (Cs+D)(s^2+4)$$

when $s = -3+5i \longrightarrow (s+3)^2+25=0$

$$60 = 0 + (C(-3+5i)+D)((-3+5i)^2+4)$$
$$= [(-3+5i)C + D][9-15i-15i-25+4]$$
$$= [(-3+5i)C + D][-12-30i]$$
$$= (-3+5i)(-12-30i)C + (-12-30i)D$$
$$= [36+90i-60i+150]C + (-12-30i)D$$
$$= (186+30i)C + (-12-30i)D$$
$$60 = (186C - 12D) + i(30C - 30D)$$

$$0 = 30C - 30D$$
$$C = D$$

$$60 = 186C - 12D$$
$$60 = 186C - 12(C)$$
$$60 = 174C$$
$$C = \frac{60}{174} = \frac{10}{29}$$

$$D = \frac{10}{29}$$

$$A = -\frac{10}{29}, \quad B = \frac{50}{29}$$

$$G(s) = \frac{As+B}{s^2+4} + \frac{Cs+D}{(s+3)^2+25}$$

$$= \frac{-\frac{10}{29}s + \frac{50}{29}}{s^2+4} + \frac{\frac{10}{29}s + \frac{10}{29}}{(s+3)^2+25}$$

$$= \frac{29 \quad 29}{s^2+4} + \frac{29 \quad 29}{(s+3)^2+25}$$

$$= \frac{1}{29}\left[\frac{-10s+50}{s^2+4}\right] + \frac{1}{29}\left[\frac{10s+10}{(s+3)^2+25}\right]$$

$$= \frac{1}{29}\left[\frac{-10s}{s^2+4} + \frac{50}{s^2+4}\right] + \frac{1}{29}\left[\frac{10(s+3)}{(s+3)^2+25} + \frac{-20}{(s+3)^2+25}\right]$$

$$= \frac{1}{29}\left[-10\left(\frac{s}{s^2+4}\right) + 25\left(\frac{2}{s^2+4}\right)\right] + \frac{1}{29}\left[10\left(\frac{s+3}{(s+3)^2+25}\right) - 4\left(\frac{5}{(s+3)^2+25}\right)\right]$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \frac{-10}{29}\cos(2t) + \frac{25}{29}\sin(2t) + \frac{10}{29}e^{-3t}\cos(5t) - \frac{4}{29}e^{-3t}\sin(5t)$$

$$\boxed{g(t) = \frac{5}{29}\left[-2\cos(2t) + 5\sin(2t)\right] + \frac{2}{29}e^{-3t}\left[5\cos(5t) - 2\sin(5t)\right]}$$