Lecture 12 (Sec 7.4)

☆ Derivatives, Integrals, & Products of Transforms:

I. Convolutions

The Laplace Transform is linear

$$\mathcal{L}\{f(t)+g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$$

$$\mathcal{L}^{-1}\{F(s)+G(s)\} = \mathcal{L}^{-1}\{F(s)\} + \mathcal{L}^{-1}\{G(s)\}$$

not true multiplication

$$\mathcal{L}^{-1}\{F(s)G(s)\} \neq \mathcal{L}^{-1}\{F(s)\}\mathcal{L}^{-1}\{G(s)\}$$

Q: What is $\mathcal{L}^{-1}\{F(s)G(s)\}$

A: $\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(\tau)g(t-\tau)\,d\tau$ ← τ is a dummy variable for integration

$$= \int_0^t f(t-\tau)g(\tau)\,d\tau$$

$$= f(t) * g(t)$$ convolution integral

Thm (Convolution Property)

$$\mathcal{L}\{f(t)*g(t)\} = \mathcal{L}\{f(t)\}\cdot\mathcal{L}\{g(t)\}$$

and conversely

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t)*g(t)$$

Ex: Find the inverse L.T. of $H(s) = \dfrac{2}{(s+1)(s-3)}$

Notice $H(s) = \underbrace{\left(\dfrac{1}{s+1}\right)}_{F(s)}\underbrace{\left(\dfrac{2}{s-3}\right)}_{G(s)}$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = e^{-t}$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = 2e^{3t}$$

$g(t) = \mathcal{L}^{-1}\{G(s)\} = 2e^{3t}$

Using the Thm

$$\mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t)$$

$$= \int_0^t f(\tau)g(t-\tau)\,d\tau$$

$$= \int_0^t e^{-\tau}\left(2e^{3(t-\tau)}\right)d\tau$$

$$= \int_0^t 2e^{-\tau+3t-3\tau}\,d\tau$$

$$= 2e^{3t}\int_0^t e^{-4\tau}\,d\tau = 2e^{3t}\left[\frac{e^{-4\tau}}{-4}\right]_0^t$$

$$= 2e^{3t}\left[\frac{e^{-4t}}{-4} - \frac{e^0}{-4}\right] = \frac{2}{4}e^{3t}\left[1-e^{-4t}\right]$$

$$\boxed{\mathcal{L}^{-1}\{H(s)\} = \frac{1}{2}\left[e^{3t} - e^{-t}\right]}$$

exercise: check this by doing partial fractions.

II. Differentiation

Recall $\mathcal{L}\{x'(t)\} = sX(s) - x(0)$

derivative in t $\xrightarrow{\mathcal{L}}$ multiplication by s

multiply by $-t$ $\xrightarrow{\mathcal{L}}$ derivative in s

Thm (Differentiation)

$$\mathcal{L}\{-t\,f(t)\} = F'(s)$$

and conversely

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{-1}{t}\mathcal{L}\{F'(s)\}$$

and furthermore

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s) \qquad n = 1, 2, 3, \ldots$$

Ex: Find L.T. of $g(t) = t^2 \sin(kt)$

$f(t) = \sin(kt)$ $\quad n = 2$

$F(s) = \mathcal{L}\{f(t)\} = \dfrac{k}{s^2+k^2}$

$$\mathcal{L}\{t^2 \sin(kt)\} = (-1)^2 F''(s) = \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{k}{s^2+k^2}\right)\right]$$

$$= \frac{d}{ds}\left[k(-1)(s^2+k^2)^{-2}(2s)\right] = \frac{d}{ds}\left((-2ks)(s^2+k^2)^{-2}\right)$$

$$= \left[(-2k)(s^2+k^2)^{-2} + (-2ks)(-2)(s^2+k^2)^{-3}(2s)\right]$$

$$= \frac{-2k(s^2+k^2)}{(s^2+k^2)^2(s^2+k^2)} + \frac{8ks^2}{(s^2+k^2)^3} = \frac{-2ks^2 - 2k^3 + 8ks^2}{(s^2+k^2)^3}$$

$$\boxed{\mathcal{L}\{t^2 \sin(kt)\} = \frac{6ks^2 - 2k^3}{(s^2+k^2)^3}}$$

Ex: Solve the IVP

$t x'' + x' + tx = 0$ $\qquad x(0) = 1, \ x'(0) = 0$

Bessel's equation of order zero

solution $\rightarrow$ Bessel function $J_0(t)$

1. Take the L.T. of both sides

$$\mathcal{L}\{x'\} = sX(s) - x(0) = sX - 1$$

$$\mathcal{L}\{x''\} = s^2X(s) - sx(0) - x'(0) = s^2X - s$$

$$\mathcal{L}\{tx''\} = -\frac{d}{ds}\left[s^2X - s\right] = -\left[2sX + s^2X' - 1\right]$$

$$= -2sX - s^2X' + 1$$

$$\mathcal{L}\{tx\} = -\frac{d}{ds}\left[X\right] = -X'$$

$$\left[-2sX - s^2X' + 1\right] + \left[sX - 1\right] + \left[-X'\right] = 0.$$

$$\left[-s^2 - 1\right]X' + \left[-2s + s\right]X + \cancel{1 - 1} = 0$$

$$(s^2+1)X' = (-s)X$$

2. Solve for $X(s)$

$$(s^2+1)X' = -sX$$

separable ODE!

sep. of vars

$$\int \frac{X'}{X} = \int \frac{-s}{s^2+1}\,ds$$

$$\ln X = \int \frac{-s\,ds}{s^2+1}$$

$u = s^2+1$

$du = 2s\,ds$

$$= -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln u = -\frac{1}{2}\ln(s^2+1)$$

$$X(s) = (s^2+1)^{-1/2} = \frac{1}{\sqrt{s^2+1}}$$

$$x(t) = J_0(t) = \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s^2+1}}\right\}$$

definition of Bessel function

III. Integration

<u>Thm</u> (Integration of Transforms)

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\,d\sigma$$

conversely

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = t\,\mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma)\,d\sigma\right\}$$

<u>Ex</u>: Find $\mathcal{L}\left\{\frac{\sinh(t)}{t}\right\}$

$f(t) = \sinh(t)$

$F(s) = \mathcal{L}\{f(t)\} = \dfrac{1}{s^2-1}$

$$\mathcal{L}\left\{\frac{\sinh(t)}{t}\right\} = \int_s^\infty \frac{1}{\sigma^2-1}\,d\sigma = \int_s^\infty \frac{d\sigma}{(\sigma-1)(\sigma+1)}$$

$$\frac{1}{(\sigma-1)(\sigma+1)} = \frac{A}{\sigma-1} + \frac{B}{\sigma+1}$$

$1 = A(\sigma+1) + B(\sigma-1)$

when $\sigma=1 \longrightarrow 1 = A(2) + 0 \rightarrow A = 1/2$

when $\sigma=-1 \longrightarrow 1 = 0 + B(-2) \rightarrow B = -1/2$

$$= \frac{1}{2}\int_s^\infty \frac{d\sigma}{\sigma-1} + \frac{-1}{2}\int_s^\infty \frac{d\sigma}{\sigma+1}$$

$$= \frac{1}{2}\Big[\ln(\sigma-1)\Big]_s^\infty - \frac{1}{2}\Big[\ln(\sigma+1)\Big]_s^\infty$$

$$= \lim_{b\to\infty} \frac{1}{2}\Big[\ln(b-1) - \ln(s-1) - \ln(b+1) + \ln(s+1)\Big]$$

$$= \lim_{b\to\infty} \frac{1}{2}\Big[\underbrace{\ln\Big(\underbrace{\frac{b-1}{b+1}}_{\to 1}\Big)}_{\to 0} + \ln\Big(\frac{s+1}{s-1}\Big)\Big]$$

$$= \boxed{\frac{1}{2}\ln\left(\frac{s+1}{s-1}\right) = \mathcal{L}\left\{\frac{\sinh(t)}{t}\right\}}$$

Sometimes its easier to use integral property of L.T. then to do partial fractions.

<u>Ex</u>: Find $\mathcal{L}^{-1}\left\{\dfrac{2s}{(s^2-1)^2}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{2s}{(s^2-1)^2}\right\} = t\,\mathcal{L}^{-1}\left\{\int_s^\infty \frac{2\sigma\,d\sigma}{(\sigma^2-1)^2}\right\} \qquad \begin{aligned} u &= \sigma^2-1 \\ du &= 2\sigma\,d\sigma \end{aligned}$$

$$\mathcal{L}^{-1}\left\{\frac{2s}{(s^2-1)^2}\right\} = t\,\mathcal{L}^{-1}\left\{\int_s \frac{2\sigma\, d\sigma}{(\sigma^2-1)^2}\right\} \qquad du = 2\sigma\, d\sigma$$

$$= t\,\mathcal{L}^{-1}\left\{\int_{s^2-1}^{\infty} \frac{du}{u^2}\right\} = t\,\mathcal{L}^{-1}\left\{\left[\frac{u^{-1}}{-1}\right]_{s^2-1}^{\infty}\right\}$$

$$= t\,\mathcal{L}^{-1}\left\{\lim_{b\to\infty}\left[\underbrace{-\frac{1}{b}}_{0} - \frac{-1}{s^2-1}\right]\right\}$$

$$= t\,\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \boxed{t\sinh(t)}$$

IV Properties:

convolution

$$\begin{cases} \mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\cdot\mathcal{L}\{g(t)\} = F(s)G(s) \\ \mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t) \end{cases}$$

differentiation

$$\begin{cases} \mathcal{L}\{-t f(t)\} = F'(s) \\ \mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s) \\ f(t) = \mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}\{F'(s)\} \end{cases}$$

integration

$$\begin{cases} \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(\sigma)\, d\sigma \\ f(t) = \mathcal{L}^{-1}\{F(s)\} = t\,\mathcal{L}^{-1}\left\{\int_s^{\infty} F(\sigma)\, d\sigma\right\} \end{cases}$$