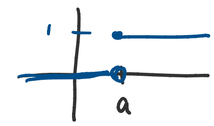
Lecture 13 (Sec 7.5)

# Periodic & Piecewise Continuous Input Functions

## I. Piecewise Continuous:

unit step function $u_a(t) = u(t-a) = \begin{cases} 0 & t<a \\ 1 & t \geq a \end{cases}$



→ flipping a switch

In Lec 9 we derived $\mathcal{L}\{u(t-a)\} = \dfrac{e^{-as}}{s}$

We can think of this as

$$\mathcal{L}\{u(t-a)(1)\} = e^{-as}\mathcal{L}\{1\} = \frac{e^{-as}}{s}$$

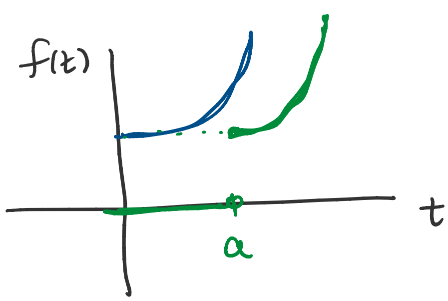
**Thm:** (Translation on the t-axis)

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$$

and conversely

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a)$$

Note: $u(t-a)f(t-a) = \begin{cases} 0 & t<a \\ f(t-a) & t \geq a \end{cases}$




"time delay of a"

Ex: Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-5s}}{s^3}\right\} = \mathcal{L}^{-1}\Big\{e^{-5s}\underbrace{\left(\frac{1}{s^3}\right)}_{F(s)}\Big\} = u(t-5)f(t-5)$

$F(s) = \dfrac{1}{s^3}$ $\quad f(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s^3}\right\} = \dfrac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{2}{s^3}\right\} = \dfrac{t^2}{2}$

$$\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s^3}\right\} = u(t-5)\frac{(t-5)^2}{2}$$

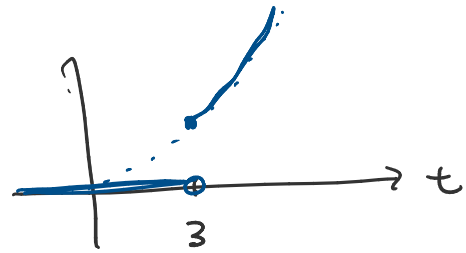
$\{\ 0 \qquad t<5$

$$\mathcal{L}\{\ \frac{\ \ }{s^3}\ \} \qquad \frac{\ \ }{2}$$

$$= \begin{cases} 0 & t < 5 \\ \frac{1}{2}(t-5)^2 & t \geq 5 \end{cases}$$

Ex: Find $\mathcal{L}\{g(t)\}$ where

$$g(t) = \begin{cases} 0 & t < 3 \\ t^2 & t \geq 3 \end{cases}$$




Need $g(t) = u(t-3)\, f(t-3)$

$f(t-3) = t^2 \qquad f(t) = (t+3)^2$

Apply the Thm:

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{u(t-3)\, f(t-3)\}$$

$$= e^{-3s}\mathcal{L}\{f(t)\} = e^{-3s}\mathcal{L}\{(t+3)^2\}$$

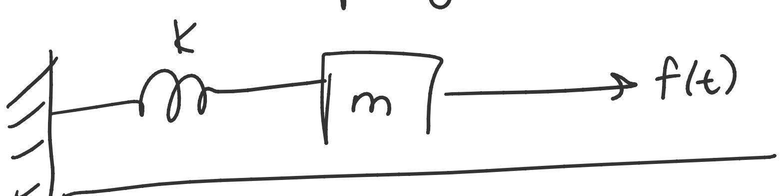
$$= e^{-3s}\mathcal{L}\{t^2 + 6t + 9\}$$

$$= e^{-3s}\left(\mathcal{L}\{t^2\} + 6\mathcal{L}\{t\} + \mathcal{L}\{9\}\right)$$

$$= e^{-3s}\left[\frac{2}{s^3} + 6\left(\frac{1}{s^2}\right) + 9\left(\frac{1}{s}\right)\right]$$

$$\boxed{= e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]}$$

Ex: Mass-on-a-spring

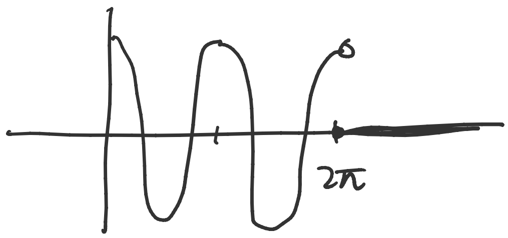



$x'' + 4x = f(t) \qquad x(0) = x'(0) = 0$

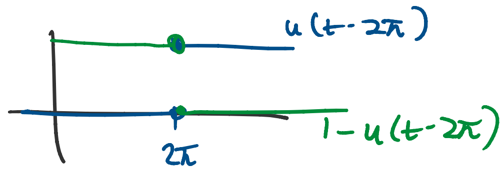
$$f(t) = \begin{cases} \cos(2t) & 0 \leq t < 2\pi \end{cases}$$

$$x'' + 4x = f(t) \quad \ldots$$

where $f(t) = \begin{cases} \cos(2t) & 0 \le t < 2\pi \\ 0 & t \geq 2\pi \end{cases}$




$$f(t) = \left[1 - u(t-2\pi)\right]\cos(2(t-2\pi))$$




$$\mathcal{L}\{f(t)\} = \mathcal{L}\{[1-u(t-2\pi)]\cos(2t)\}$$

$$= \mathcal{L}\{\cos(2t)\} - \mathcal{L}\{u(t-2\pi)\overbrace{\cos(2(t-2\pi))}^{f(t-2\pi)}\}$$

$$= \frac{s}{s^2+4} - e^{-2\pi s}F(s)$$

$$= \frac{s}{s^2+4} - e^{-2\pi s}\frac{s}{s^2+4} = \frac{s(1-e^{-2\pi s})}{s^2+4}$$

Let's take L.T. of ODE

$$x'' + 4x = f(t) \qquad x(0) = x'(0) = 0$$

$$(s^2X - s\cdot 0 - 0) + 4X = \frac{s(1-e^{-2\pi s})}{s^2+4}$$

$$(s^2+4)X = \frac{s(1-e^{-2\pi s})}{s^2+4}$$

$$X(s) = \frac{s(1-e^{-2\pi s})}{(s^2+4)^2} = \frac{s}{(s^2+4)^2} - \frac{e^{-2\pi s}s}{(s^2+4)^2}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}s}{(s^2+4)^2}\right\}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}\, s}{(s^2+4)^2}\right\}$$

$$e^{-2\pi s}\; F(s)$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$$ fcn is easy to integrate
use integration

$$= t\,\mathcal{L}^{-1}\left\{\int_s^{\infty} \frac{\sigma}{(\sigma^2+4)^2}\,d\sigma\right\} \qquad u = \sigma^2+4, \quad du = 2\sigma\, d\sigma$$

$$= t\,\mathcal{L}^{-1}\left\{\frac{1}{2}\int_{s^2+4}^{\infty} \frac{du}{u^2}\right\}$$

$$= t\,\mathcal{L}^{-1}\left\{\frac{1}{2}\left[\frac{u^{-1}}{-1}\right]_{s^2+4}^{\infty}\right\}$$

$$= t\,\mathcal{L}^{-1}\left\{\frac{1}{2}\lim_{b\to\infty}\left[\underbrace{\frac{-1}{b}}_{\to 0} - \frac{-1}{s^2+4}\right]\right\}$$

$$= t\,\mathcal{L}^{-1}\left\{\frac{1}{2}\cdot\frac{1}{s^2+4}\right\} = \frac{t}{4}\,\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \underline{\frac{t}{4}\sin(2t)}$$

then

$$\mathcal{L}^{-1}\left\{e^{-2\pi s}\underbrace{\left(\frac{s}{(s^2+4)^2}\right)}_{G(s)}\right\} = u(t-2\pi)\,g(t-2\pi)$$

$$= u(t-2\pi)\,\frac{(t-2\pi)}{4}\sin(2(t-2\pi))$$

$$= u(t-2\pi)\,\frac{(t-2\pi)}{4}\sin(2t)$$

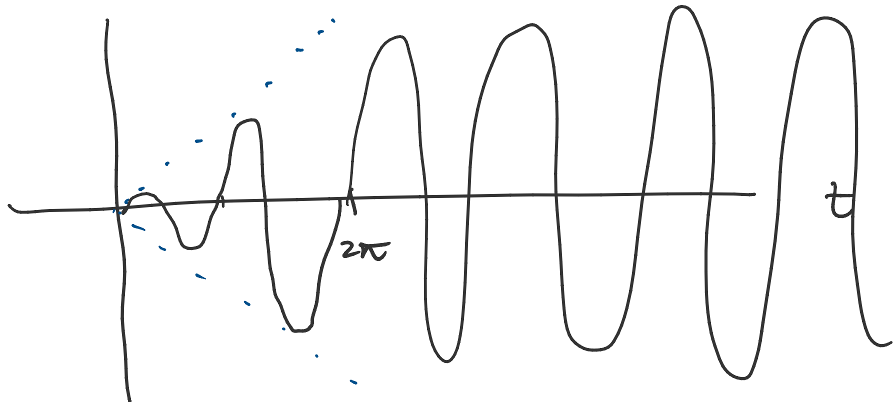
So

$$x(t) = \frac{t}{4}\sin(2t) - \frac{1}{4}u(t-2\pi)(t-2\pi)\sin(2t)$$

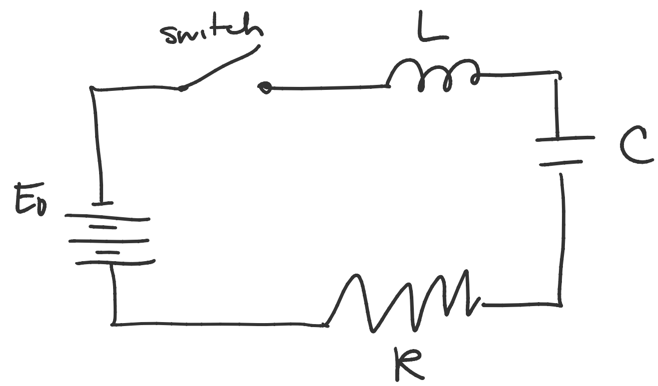
$$= \frac{1}{4}\left[t - u(t-2\pi)(t-2\pi)\right]\sin(2t)$$

$$= \begin{cases} \frac{1}{4} t \sin(2t) & t < 2\pi \\ \frac{2\pi}{4} \sin(2t) & t \geq 2\pi \end{cases}$$

plot




Ex: RLC circuit




$R = 110\Omega$

$L = 1H$

$C = 0.001F$

$E_0 = 90V$

switch on for $0 < t < 1$

switch off for $t > 1$

circuit eqns:

i – current

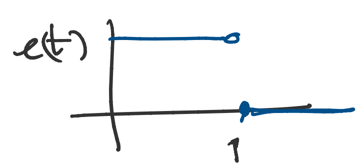
q – charge

$$L\frac{di}{dt} + Ri + \frac{1}{C}q = e(t)$$

$i(0) = 0$

$$\frac{di}{dt} + 110i + 1000q = e(t)$$

$$e(t) = \begin{cases} 90 & 0 < t < 1 \\ 0 & t > 1 \end{cases} = 90[1 - u(t-1)]$$




Recall $i = \frac{dq}{dt}$, could write

$$\frac{d^2i}{} + 110\cancel{\frac{di}{}} + 10\cancel{0}0i = e'(t)$$

$$\cancel{\frac{d^2i}{dt^2} + 110\frac{di}{dt} + 1000i = e'(t)}$$

~~but~~ $\cancel{e'(t) = \begin{Bmatrix} 0 & t \neq 1 \\ \text{undef} & t = 1 \end{Bmatrix}}$

Instead $\quad q(t) = \int_0^t i(\tau)d\tau$

$$\frac{di}{dt} + 110i + 1000\int_0^t i(\tau)d\tau = 90[1-u(t-1)]$$

$$i(0) = 0$$

integrodifferential equation

With L.T. $\quad \mathcal{L}\left\{\int_0^t i(\tau)d\tau\right\} = \frac{I(s)}{s}$

Take L.T. of IDE

$$sI + 110I + 1000\frac{I}{s} = 90\left[\frac{1}{s} - \frac{e^{-s}}{s}\right]$$

$$\left(\frac{s^2 + 110s + 1000}{s}\right)I = \frac{90}{s}\left[1 - e^{-s}\right]$$

$$I(s) = \frac{90(1-e^{-s})}{s^2 + 110s + 1000} = \frac{90(1-e^{-s})}{(s+10)(s+100)}$$

partial fractions:

$$\frac{90}{(s+10)(s+100)} = \frac{A}{s+10} + \frac{B}{s+100}$$

$$90 = A(s+100) + B(s+10)$$

when $s = -10$

$$90 = A(90) \quad \longrightarrow \quad A = 1$$

when $s = -100$
$$90 = 0 + B(-90) \rightarrow B = -1$$

$$\frac{90}{(s+10)(s+100)} = \frac{1}{s+10} - \frac{1}{s+100}$$

$$I(s) = \left[1 - e^{-s}\right]\left[\frac{1}{s+10} - \frac{1}{s+100}\right]$$

$$= \underbrace{\left[\frac{1}{s+10} - \frac{1}{s+100}\right]}_{F(s)} - e^{-s}\left[\frac{1}{s+10} - \frac{1}{s+100}\right]$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+10} - \frac{1}{s+100}\right\} = e^{-10t} - e^{-100t}$$

$$i(t) = \left(e^{-10t} - e^{-100t}\right) - u(t-1)\left[e^{-10(t-1)} - e^{-100(t-1)}\right]$$

$$= \begin{cases} e^{-10t} - e^{-100t} & 0 < t < 1 \\ e^{-10t} - e^{-100t} - e^{-10(t-1)} + e^{-100(t-1)} & t \geq 1 \end{cases}$$

## II. Periodic Functions

**Def** A function $f(t)$ is periodic if there is a number $p > 0$ such that

$$f(t+p) = f(t) \quad \text{for all } t > 0$$

$p$ is called the period of fcn $f(t)$

**Thm** (Transforms of Periodic Fcns)

... periodic with period $p$ and

<u>Thm</u> (Transforms of Periodic Fns)

If $f(t)$ is periodic with period $p$ and piecewise continuous for $t \geq 0$, then

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt$$

<u>proof</u>:

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \sum_{n=0}^{\infty} \int_{np}^{(n+1)p} e^{-st} f(t)\,dt$$ break up integral over each period

Do a change of variables $t = \tau + np$ $dt = d\tau$

$$= \sum_{n=0}^{\infty} \int_0^p \underbrace{e^{-s(\tau+np)}}_{\substack{\text{pull out} \\ \text{const. term}}} \underbrace{f(\tau+np)}_{\text{use periodicity}} d\tau$$

$$= \sum_{n=0}^{\infty} e^{-nps} \underbrace{\int_0^p e^{-s\tau} f(\tau)\,d\tau}_{\text{pull this out of the sum}}$$

$$= \underbrace{\left( \sum_{n=0}^{\infty} e^{-nps} \right)}\int_0^p e^{-s\tau} f(\tau)\,d\tau$$

Let $x = e^{-ps}$ Note that $|x| < 1$

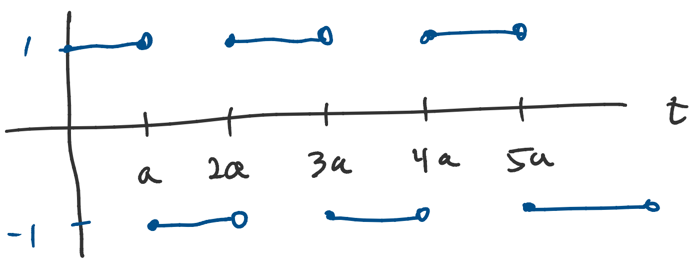
then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ this is a power series

so $$F(s) = \left( \frac{1}{1-e^{-ps}} \right) \int_0^p e^{-st} f(t)\,dt$$

so $$F(s) = \left(\frac{1}{1-e^{-ps}}\right)\int_0^p e^{-st} f(t)\,dt$$ $\square$

<u>Ex</u>: $f(t) = (-1)^{\lfloor \frac{t}{a} \rfloor}$

$\lfloor x \rfloor = \text{floor}(x)$
greatest integer $\leq x$
$\lfloor 1.5 \rfloor = 1$ $\quad$ $\lfloor 1.999 \rfloor = 1$




period $p = 2a$

$$\mathcal{L}\left\{(-1)^{\lfloor \frac{t}{a} \rfloor}\right\} = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st}(-1)^{\lfloor \frac{t}{a} \rfloor}\,dt$$

$$= \left(\frac{1}{1-e^{-2as}}\right)\left[\int_0^a e^{-st}(1)\,dt + \int_a^{2a} e^{-st}(-1)\,dt\right]$$

$$= \left(\frac{1}{1-e^{-2as}}\right)\left[\left(\frac{e^{-st}}{-s}\right)\Bigg|_0^a + \left(\frac{-e^{-st}}{-s}\right)\Bigg|_a^{2a}\right]$$

$$= \left(\frac{1}{1-e^{-2as}}\right)\left[\frac{e^{-as}}{-s} - \frac{1}{-s} + \frac{-e^{-2as}}{-s} - \frac{-e^{-as}}{-s}\right]$$

$$= \left(\frac{1}{1-e^{-2as}}\right)\left[\frac{e^{-2as} - 2e^{-as} + 1}{s}\right]$$

$$= \frac{(1-e^{-as})^{\cancel{2}}}{s\,\cancel{(1-e^{-as})}(1+e^{-as})} = \frac{1-e^{-as}}{s(1+e^{-as})}\cdot\frac{e^{as/2}}{e^{as/2}}$$

$$= \frac{e^{as/2} - e^{-as/2}}{s\left(e^{as/2} + e^{as/2}\right)} = \frac{\cancel{2}\sinh\left(\frac{as}{2}\right)}{s\,\cancel{2}\cosh\left(\frac{as}{2}\right)} = \boxed{\frac{1}{s}\tanh\left(\frac{as}{2}\right)}$$