☆ Impulses & Delta Functions

I. Delta Functions:

Motivation — impulsive force — acts during a short time

e.g. – bat striking a ball
– surge of voltage

The effect depends on:

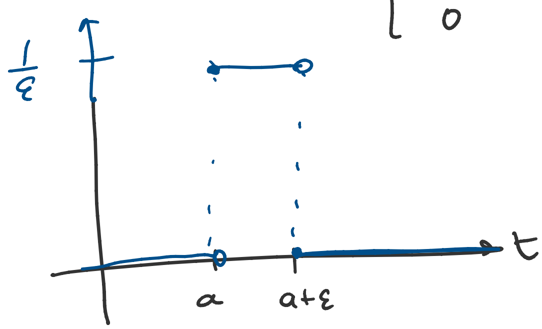
$$p = \int_a^b f(t)\,dt$$

the impulse of the force $f(t)$ over the interval $[a,b]$

Q: How do we model this?

Def: The unit impulse $d_{a,\varepsilon}(t)$ is

$$d_{a,\varepsilon}(t) = \begin{cases} \frac{1}{\varepsilon} & \text{if } a \le t < a+\varepsilon \\ 0 & \text{otherwise} \end{cases}$$



as $\varepsilon \to 0$
$\frac{1}{\varepsilon} \to \infty$

$$p = \int_0^\infty d_{a,\varepsilon}(t)\,dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\,dt = \varepsilon \cdot \frac{1}{\varepsilon} = 1$$

Q: What about instantaneous

$$\delta_a(t) = \lim_{\varepsilon \to 0} d_{a,\varepsilon}(t) = \lim_{\varepsilon \to 0} \begin{Bmatrix} \begin{cases} 1/\varepsilon & a \le t < a+\varepsilon \\ 0 & \text{else} \end{cases} \end{Bmatrix}$$

$$\delta_a(t) = \begin{cases} +\infty & \text{if } t = a \\ 0 & \text{if } t \ne a \end{cases}$$

$$\delta_a(t) = \begin{cases} +\infty & \text{if } \\ 0 & \text{if } t \neq a \end{cases}$$

$$P = \int_0^\infty \delta_a(t)\,dt = \lim_{\varepsilon \to 0} \int_0^\infty d_{a,\varepsilon}(t)\,dt = \lim_{\varepsilon \to 0} 1 = 1$$

Def: The Dirac delta function is $\delta_a(t)$

$$\delta_a(t) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{if } t \neq a \end{cases} \quad \text{and} \quad \int_0^\infty \delta_a(t)\,dt = 1$$

Think of it as an operator

$$\int_0^\infty g(t)\,\delta_a(t)\,dt = \underset{\uparrow}{g(a)}$$

picks out the value at $t = a$

$$\underset{\text{function}}{g(t)} \longrightarrow \boxed{\int_0^\infty \cdot\ \delta_a(t)\,dt} \longrightarrow \underset{\text{number}}{g(a)}$$

Ex:

$$\underbrace{\int_0^\infty e^{-st}\,\delta_a(t)\,dt}_{\text{this is definition of the L.T.}} = e^{-as}$$

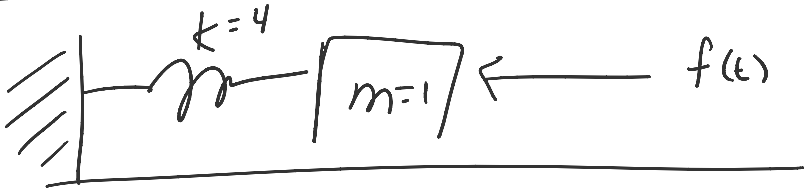
$$\boxed{\mathcal{L}\{\delta_a(t)\} = e^{-as}} \qquad a > 0$$

Note: we call $\delta(t) = \delta_0(t)$ the Dirac delta at 0

$$\mathcal{L}\{\delta(t)\} = e^{-0 \cdot s} = 1$$

$$\boxed{\mathcal{L}\{\delta(t)\} = 1}$$

Ex: Mass on a spring



at $t=2\pi$ the mass is struck by a hammer impulse $p=8$

$$x'' + 4x = 8\,\delta_{2\pi}(t) = 8\,\delta(t-2\pi) \qquad x(0)=3 \quad x'(0)=0$$

Take the L.T. of both sides

$$(s^2X - 3\cdot s - 0) + 4X = 8\mathcal{L}\{\delta_{2\pi}(t)\} = 8e^{-2\pi s}$$

$$(s^2+4)X - 3s = 8e^{-2\pi s}$$

$$(s^2+4)X = 3s + 8e^{-2\pi s}$$

$$X(s) = \frac{3s}{s^2+4} + \frac{8e^{-2\pi s}}{s^2+4}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{3s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{8e^{-2\pi s}}{s^2+4}\right\} \qquad F(s)=\frac{2}{s^2+4} \quad f(t)=\sin(2t)$$

$$= 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 4\mathcal{L}^{-1}\left\{\left(\frac{2}{s^2+4}\right)e^{-2\pi s}\right\}$$

$F(s)\,e^{-2\pi s}$ ← translation on the t-axis

$$= 3\cos(2t) + 4u(t-2\pi)\sin(2(t-2\pi))$$

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$$= \begin{cases} 3\cos(2t) & t < 2\pi \\ 3\cos(2t) + 4\sin(2t) & t \geq 2\pi \end{cases}$$

We can simplify the 2nd term

$$3\cos(2t) + 4\sin(2t) = R\cos(2t - \alpha)$$

$$R = \sqrt{3^2+4^2} = 5$$

$$\alpha = \tan^{-1}(4/3) \approx 0.9273$$

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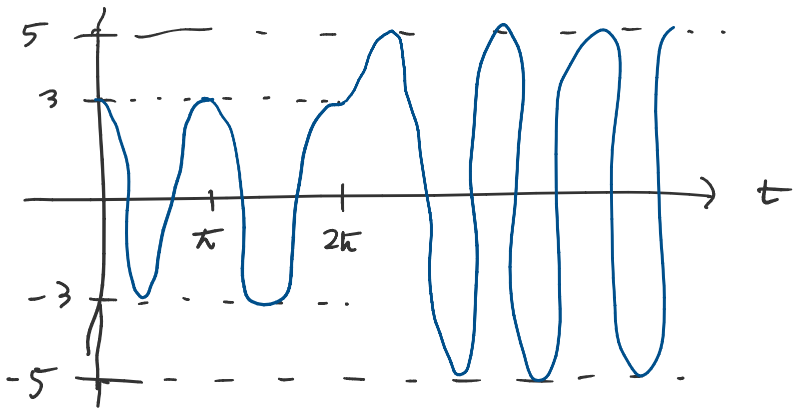
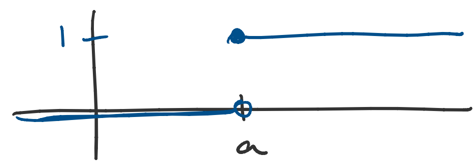
$$x(t) = \begin{cases} 3\cos(2t) & t < 2\pi \\ 5\cos(2t - \alpha) & t \geq 2\pi \end{cases}$$



@ $t = 2\pi$ instantaneously the amplitude increases from 3 to 5.

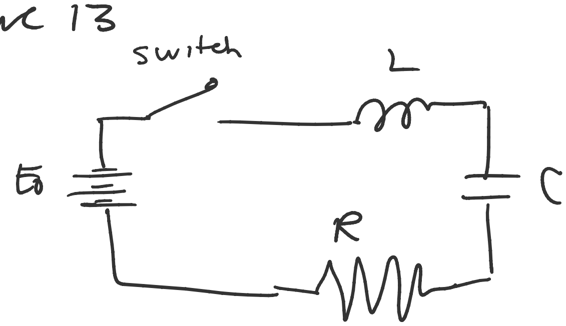
<u>Note:</u> $\frac{d}{dt} u_a(t) = \begin{cases} 0 & t < a \\ \infty & t = a \\ 0 & t > a \end{cases} = \delta_a(t)$

$$u_a(t) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$



$\delta_a(t)$ is the derivative of $u_a(t)$

<u>Ex:</u> RLC circuit in Lecture 13



$$L\frac{di}{dt} + Ri + \frac{1}{C} q = e(t)$$

$$i = \frac{dq}{dt}$$

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C} i = e'(t)$$

$L = 1$ $R = 110$ $C = 0.001$

$$e(t) = 90 - 90\,u(t-1)$$

$$e'(t) = -90\,\underline{d}\,u(t-1) = -90\,\delta_1(t)$$

$e(t) - \cdots$

$$e'(t) = -90 \frac{d}{dt} u(t-1) = -90\, \delta_1(t)$$

<u>Now</u>: $i'' + 110 i' + 1000 i = -90\, \delta_1(t) \qquad i(0)=0 \quad i'(0)=90$

Take L.T. of both sides

$$(s^2 I - 0 \cdot s - 90) + 110(sI - 0) + 1000 I = -90 e^{-s}$$

$$(s^2 + 110 s + 1000) I - 90 = -90 e^{-s}$$

$$\boxed{I(s) = \frac{90(1 - e^{-s})}{(s+10)(s+100)}}$$

this is the same as in Lec 13

II. Duhamel's Principle

Consider a physical system

$$a x'' + b x' + c x = f(t) \qquad x(0) = x'(0) = 0$$

$x(t)$ — output or response

$f(t)$ — input

Take L.T. of both sides

$$a s^2 X + b s X + c X = F(s)$$

$$X(s) = \frac{F(s)}{a s^2 + b s + c} = W(s) F(s)$$

$$W(s) = \frac{1}{a s^2 + b s + c}$$ <u>transfer function</u> of the system

$$w(t) = \mathcal{L}^{-1}\{W(s)\}$$ <u>weight function</u>

use convolution property

use convolution:

$$x(t) = \mathcal{L}^{-1}\{W(s)F(s)\} = w(t) * f(t)$$

$$\boxed{x(t) = \int_0^t w(\tau) f(t-\tau)\, d\tau}$$

Duhamel's Principle

Key step: $w(t) = \mathcal{L}^{-1}\{W(s)\}$

Ex: $x'' + 6x' + 10x = f(t)$ $\quad x(0) = x'(0) = 0$

$$s^2 X + 6sX + 10X = F(s)$$

$$X(s) = \frac{F(s)}{s^2+6s+10}$$

$$W(s) = \frac{1}{s^2+6s+10} = \frac{1}{(s+3)^2+1}$$

$$w(t) = \mathcal{L}^{-1}\{W(s)\} = e^{-3t}\sin(t)$$

So, By Duhamel's Principle

$$\boxed{x(t) = \int_0^t e^{-3\tau}\sin(\tau) f(t-\tau)\, d\tau}$$