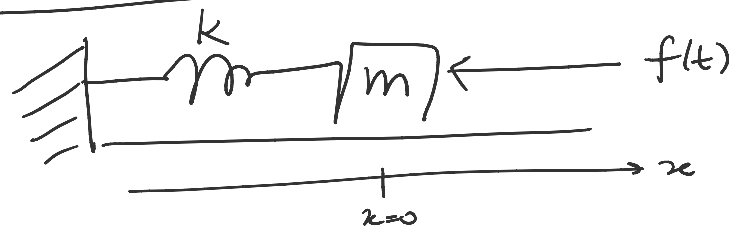
Lecture 15 (Sec 9.1)

Chapter 9: Fourier Series Methods and PDEs

☆ 9.1: Periodic Functions & Trigonometric Series

I. Periodic Functions:



$$x'' + w_0^2 x = f(t) \quad \text{where } w_0^2 = \frac{k}{m}$$

solve using method of undetermined coefr.

if $f(t) = A\cos(wt)$

and $w_0^2 \neq w^2$ then

$$x(t) = \frac{A}{w_0^2 - w^2}\cos(wt)$$

Suppose $f(t) = \sum_{n=1}^{N} A_n \cos(w_n t)$

(typo in video)

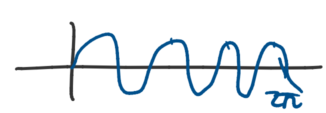
If $w_n^2 \neq w_0^2$, then

$$x_p(t) = \sum_{n=1}^{N} \frac{A_n}{w_0^2 - w_n^2}\cos(w_n t)$$

Note: many mechanical + electrical systems involve periodic forcing fcns $f(t)$.
If $f(t)$ is reasonably "nice" we can represent our solutions as series of trigonometric functions

Def: The function $f(t)$ is periodic provided there exists $p > 0$ such that $f(t+p) = f(t)$ for all t.
The value p is called the period of f.

Ex: $g(t) = \cos(3t)$ period $p = \frac{2\pi}{3}$



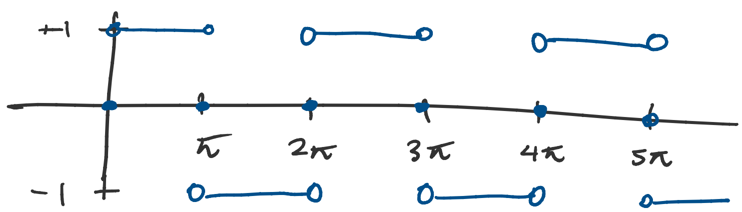
Ex: $g(t) = \cos(3t)$ period $p = \frac{2\pi}{3}$ 

$$\cos\left(3\left(t + \tfrac{2\pi}{3}\right)\right) = \cos(3t + 2\pi) = \cos(3t)$$

Ex: $f(t) = 3 + \cos(t) - \sin(t) + 5\cos(2t) + 17\sin(3t)$

Since $\cos(nt)$ and $\sin(nt)$ where $n = 1, 2, 3, \dots$
all have period of 2π
so $f(t)$ will also have a period of 2π.

Ex: Consider the square wave:



has period
$p = 2\pi$

II. Fourier Series of period 2π functions

In 1822, Joseph Fourier asserted that every fcn $f(t)$ with period 2π can be represented by an <u>infinite</u> trigonometric series

$$f(t) \sim \underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt)\right)}_{\text{a series of this form is called a Fourier series}}$$

true with some mild restrictions on $f(t)$

Def: two functions $u(t)$ and $v(t)$ are <u>orthogonal</u> on the interval $[a, b]$ if

$$\int_a^b u(t)v(t)\,dt = 0$$

Note: $\cos(nt)$ and $\sin(nt)$ $n = 1, 2, 3, \dots$
are orthogonal on $[-\pi, \pi]$

$\int^{\pi}$ for all m, n

are orthogonal on $[-\pi, \pi]$

$$\int_{-\pi}^{\pi} \cos(mt)\sin(nt)\,dt = 0 \qquad \text{for all } m,n$$

$$\int_{-\pi}^{\pi} \cos(mt)\cos(nt)\,dt = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mt)\sin(nt)\,dt = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases}$$

Now, let's assume Fourier's assertion is true

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(mt) + b_m \sin(mt) \right)$$

We will derive the values of a_m and b_m.

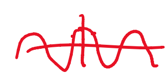
Derive a_0: integrate both sides over $[-\pi, \pi]$

$$\int_{-\pi}^{\pi} f(t)\,dt = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dt + \sum_{m=1}^{\infty} \left(a_m \underbrace{\int_{-\pi}^{\pi} \cos(mt)\,dt}_{0} \right) + \sum_{m=1}^{\infty} \left(b_m \underbrace{\int_{-\pi}^{\pi} \sin(mt)\,dt}_{0} \right)$$

$\sin(-x) = -\sin(x)$

$$\int_{-\pi}^{\pi} \cos(mt)\,dt = \left[\frac{\sin(mt)}{m} \right]_{-\pi}^{\pi} = \frac{\sin(m\pi)}{m} - \frac{\sin(-m\pi)}{m}$$

$$= \frac{\sin(m\pi) + \sin(m\pi)}{m} = \frac{2\sin(m\pi)}{m} = 0$$



$\cos(-x) = \cos(x)$

$$\int_{-\pi}^{\pi} \sin(mt)\,dt = \left[-\frac{\cos(mt)}{m} \right]_{-\pi}^{\pi} = -\frac{\cos(m\pi)}{m} + \frac{\cos(-m\pi)}{m}$$

$$= \frac{-\cos(m\pi) + \cos(m\pi)}{m} = 0$$

$$\int_{-\pi}^{\pi} f(t)\,dt = \frac{a_0}{2} \int_{-\pi}^{\pi} dt = \frac{a_0}{2} \left[t \right]_{-\pi}^{\pi} = \frac{a_0}{2} \left[\pi - (-\pi) \right] = \frac{\cancel{2}\pi a_0}{\cancel{2}}$$

$$\boxed{a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\,dt}$$

$\frac{a_0}{2}$ is the average value of $f(t)$ over $[-\pi, \pi]$

as we add more terms in the series, it will get ... to $f(t)$.

as we add more ... in the series, it will get closer to $f(t)$.

Now let's derive a_m, use the fact that $\cos(mt)$ terms are orthogonal

$$\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \frac{a_0}{2}\underbrace{\cancel{\int_{-\pi}^{\pi}\cos(nt)\,dt}}_{0} + \sum_{n=1}^{\infty} a_m\left(\underbrace{\int_{-\pi}^{\pi}\cos(mt)\cos(nt)\,dt}_{\begin{cases}0 & \text{if } n\neq m\\ \pi & \text{if } n=m\end{cases}}\right)$$

$$+\sum_{n=1}^{\infty} b_m\left(\underbrace{\cancel{\int_{-\pi}^{\pi}\sin(mt)\cos(nt)\,dt}}_{0}\right)$$

$$\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = a_n\pi \qquad \boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt}$$

Similarly, we can derive b_m, by integrate against $\sin(nt)$

$$\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \frac{a_0}{2}\underbrace{\cancel{\int_{-\pi}^{\pi}\sin(nt)\,dt}}_{0} + \sum_{m=1}^{\infty} a_m\left(\underbrace{\cancel{\int_{-\pi}^{\pi}\cos(mt)\sin(nt)\,dt}}_{0,\ \text{orthogonal}}\right)$$

$$+\sum_{m=1}^{\infty} b_m\left(\underbrace{\int_{-\pi}^{\pi}\sin(mt)\sin(nt)\,dt}_{\text{orthogonal}\ \begin{cases}0 & \text{if } n\neq m\\ \pi & \text{if } n=m\end{cases}}\right)$$

$$= b_n\pi \qquad \boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt}$$

think of this as a "projection" like in linear algebra.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nt) + b_n\sin(nt)\right)$$

$\sin(nt)$ and $\cos(nt)$ as basis "vectors", and

$\int_{-\pi}^{\pi} u(t)v(t)\,dt$ is like a dot product of two "vectors"

Def: Let $f(t)$ be a piecewise continuous function of period 2π. Then the Fourier series of $f(t)$ is:

Def: Let $f(t)$ be a piece...
period 2π. Then the <u>Fourier series</u> of $f(t)$ is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt) \right)$$

where the <u>Fourier coefficents</u> a_n and b_n are

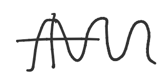
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt \qquad n = 1, 2, 3, \ldots$$

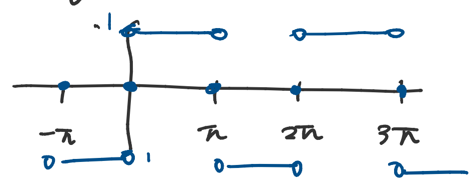
<u>Note</u>: Sometimes this series fails to converge to the fcn $f(t)$ at some points t.

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt) \right)$$

$\uparrow$ instead of $=$

<u>Ex</u>: Find the Fourier series of the square wave 

$$f(t) = \begin{cases} -1 & \text{if } -\pi < t < 0 \\ +1 & \text{if } 0 < t < \pi \\ 0 & \text{if } t = -\pi, 0, \pi \end{cases}$$



Let's calculate a_0

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt = \frac{1}{\pi} \left[\int_{-\pi}^{0} (-1)\, dt + \int_{0}^{\pi} (+1)\, dt \right]$$

$$= \frac{1}{\pi} \left[(-t)\Big|_{-\pi}^{0} + (t)\Big|_{0}^{\pi} \right] = \frac{1}{\pi} \left[-0 - (-(-\pi)) + (\pi - 0) \right]$$

$$= \frac{1}{\pi} \left[-\pi + \pi \right] = 0 \qquad \boxed{a_0 = 0}$$ the average value of $f(t)$ is zero

Now, let's calculate a_n

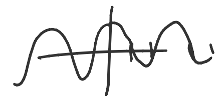
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt$$

$$= \ldots \int_{-\pi}^{0} \ldots dt + \int_{0}^{\pi} (+1) \cos(nt)\, dt \Big]$$

$$a_n = \frac{1}{\pi}\int_{-\pi} f(t)\cos(nt)\,dt$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\cos(nt)\,dt + \int_{0}^{\pi}(+1)\cos(nt)\,dt\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{\sin(nt)}{n}\right)_{-\pi}^{0} + \left(\frac{\sin(nt)}{n}\right)_{0}^{\pi}\right]$$



$\sin(n\pi)=0$

$$= \frac{1}{\pi}\left[-\cancel{\frac{\sin(0)}{n}}^{\,0} - \left(-\cancel{\frac{\sin(-n\pi)}{n}}^{\,0}\right) + \cancel{\frac{\sin(n\pi)}{n}}^{\,0} - \cancel{\frac{\sin(0)}{n}}^{\,0}\right]$$

$$= 0 \qquad \boxed{a_n = 0}$$

expect this, since the square wave is an odd function, and $\cos(nt)$ is even.

Now let's calculate b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin(nt)\,dt + \int_{0}^{\pi}(+1)\sin(nt)\,dt\right]$$

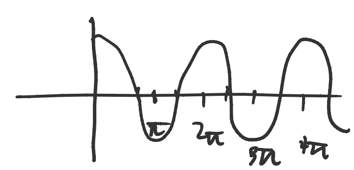
$$= \frac{1}{\pi}\left[\left(\frac{\cos(nt)}{n}\right)_{-\pi}^{0} + \left(-\frac{\cos(nt)}{n}\right)_{0}^{\pi}\right]$$

$\cos(-x) = \cos(x)$

$$= \frac{1}{\pi}\left[\frac{\overbrace{\cos(0)}^{=1}}{n} - \frac{\cos(-n\pi)}{n} + -\frac{\cos(n\pi)}{n} + \frac{\overbrace{\cos(0)}^{=1}}{n}\right]$$

$$= \frac{1}{n\pi}\left[1 - \cos(n\pi) - \cos(n\pi) + 1\right]$$

$$= \frac{1}{n\pi}\left[2 - 2\cos(n\pi)\right]$$



$$\cos(n\pi) = \begin{cases} +1 & n = \text{even} \\ -1 & n = \text{odd} \end{cases} = (-1)^n$$

$$= \frac{1}{n\pi}\left[2 - 2(-1)^n\right]$$

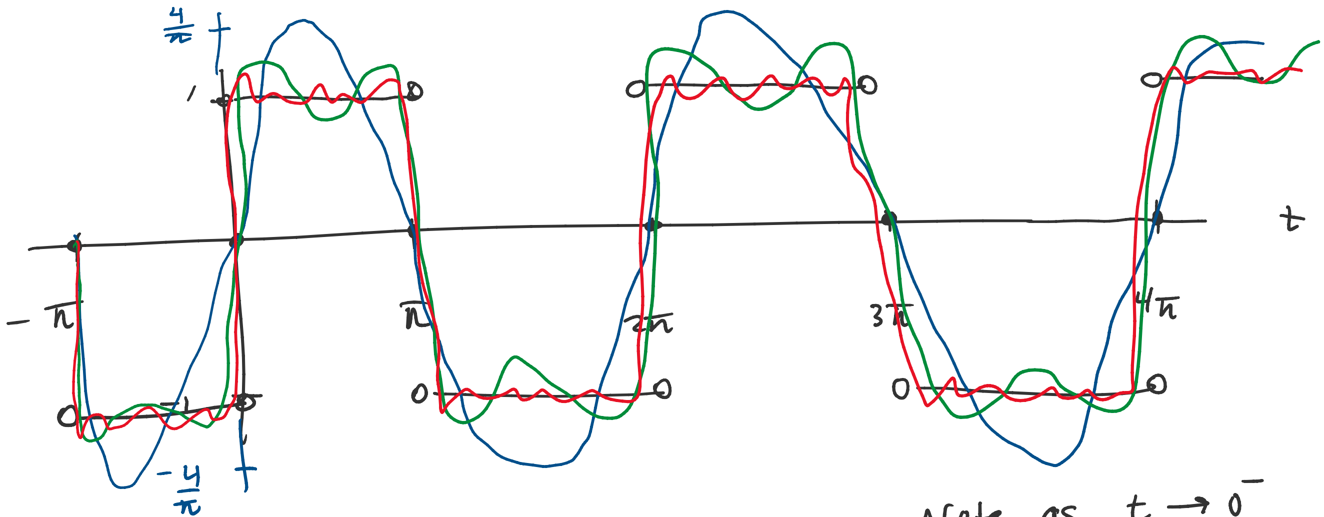
$$= \frac{2}{n\pi}\left[1 - (-1)^n\right]$$

$$b_n = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

So we can write the Fourier series

$$f(t) \sim \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n}$$

$$= \frac{4}{\pi}\left[\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \dots\right]$$



$\frac{4}{\pi}\sin(t)$

$\frac{4}{\pi}\left[\sin(t) + \frac{1}{3}\sin(3t)\right]$

$\frac{4}{\pi}\left[\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t)\right]$

Note as $t \to 0^-$ the series overshoots the value -1

near a discontinuity we see this behavior

Gibb's phenomenon

Note: When $f(t)$ has terms that are a polynomial in t, the following formulas are useful

$$\int u\cos(u)\,du = \cos(u) + u\sin(u) + C$$

$$\int u\sin(u)\,du = \sin(u) - u\cos(u) + C$$

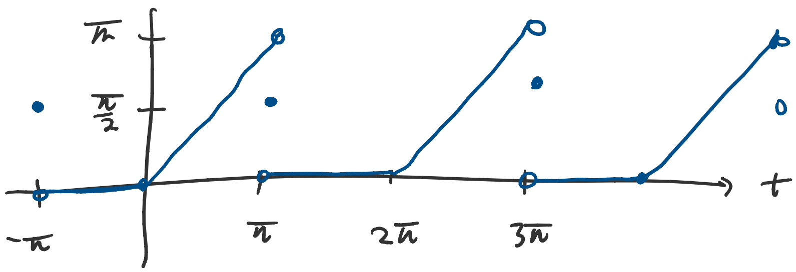
$$\int u^n\cos(u)\,du = u^n\sin(u) - n\int u^{n-1}\sin(u)\,du$$

$$\int u^n\sin(u)\,du = -u^n\cos(u) + n\int u^{n-1}\cos(u)\,du$$

derive all of these Integration by Parts

$\underline{\text{Ex}}$: Find the Fourier series of the 2π periodic fcn that is defined on one period to be

$$f(t) = \begin{cases} 0 & \text{if } -\pi < t \le 0 \\ t & \text{if } 0 \le t < \pi \\ \frac{\pi}{2} & \text{if } t = \pm\pi \end{cases}$$



Calculate a_0 first

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\left[\cancel{\int_{-\pi}^{0} 0\,dt}^{\,0} + \int_0^{\pi} t\,dt\right]$$

$$= \frac{1}{\pi}\left(\frac{t^2}{2}\right)_0^{\pi} = \frac{1}{\pi}\left(\frac{\pi^2}{2} - 0\right) = \frac{\pi}{2} \qquad \boxed{a_0 = \frac{\pi}{2}}$$

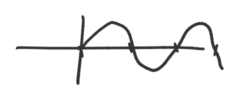
Calculate the a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \frac{1}{\pi}\int_0^{\pi} t\cos(nt)\,dt$$

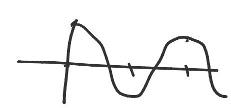
$u = nt \qquad t = \frac{u}{n} \qquad dt = \frac{du}{n}$

$$= \frac{1}{n^2\pi}\left[\int_0^{n\pi} u\cos(u)\,du\right] \qquad \int u\cos(u)\,du = \cos(u) + u\sin(u)$$

$$= \frac{1}{n^2\pi}\Big[\cos(u) + u\sin(u)\Big]_0^{n\pi}$$



$$= \frac{1}{n^2\pi}\Big[\underset{\substack{\parallel \\ (-1)^n}}{\cos(n\pi)} - \underset{\substack{\parallel \\ 1}}{\cos(0)} + \underbrace{n\pi\sin(n\pi)}_{\substack{\parallel \\ 0}} - 0\Big]$$



$$= \frac{1}{n^2\pi}\Big[(-1)^n - 1\Big] = \boxed{\begin{cases} \frac{-2}{n^2\pi} & n \text{ odd} \\ 0 & n \text{ even}\end{cases} = a_n}$$

Calculate the b_n

π $\qquad u = nt$

Calculate the b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_0^{\pi} t\sin(nt)\,dt$$

$u = nt$

$t = \frac{u}{n}$ $dt = \frac{du}{n}$

$\int u\sin(u)\,du = \sin(u) - u\cos(u)$

$$= \frac{1}{n^2\pi}\int_0^{n\pi} u\sin(u)\,du$$

$$= \frac{1}{n^2\pi}\Big[\sin(u) - u\cos(u)\Big]_0^{n\pi}$$

$$= \frac{1}{n^2\pi}\Big[\cancel{\sin(n\pi)}^{\,0} - \cancel{\sin(0)}^{\,0} - \big(n\pi\underbrace{\cos(n\pi)}_{(-1)^n} - 0\big)\Big]$$

$$= \frac{1}{n^2\pi}\Big[-n\pi(-1)^n\Big] = \frac{\cancel{n\pi}(-1)^{n+1}}{n^{\cancel{2}}\cancel{\pi}} = \boxed{\frac{(-1)^{n+1}}{n} = b_n}$$

So we write the Fourier series

$$\boxed{f(t) \sim \frac{\pi}{4} - \frac{2}{\pi}\sum_{n\ \text{odd}} \frac{\cos(nt)}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(nt)}{n}}$$

Note: If $f(t)$ is periodic with $p = 2\pi$, then we can also compute the Fourier coefficients by integrating over $[0, 2\pi]$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\,dt$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\,dt$$