Lecture 16 (Sec 9.2)

☆ General Fourier Series & Convergence

I. General Fourier Series:

In Lec 15 – F.S. for $f(t)$ with $p = 2\pi$

Define for $f(t)$ with period $P = 2L$

L is called the half-period

Def: Let $f(t)$ be a piecewise continuous function of period $2L$ that is defined for all t. Then the Fourier Series of $f(t)$ is

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right)$$

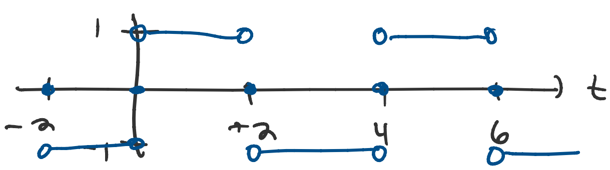
where the Fourier coefficients are

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right) dt$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) dt$$

Ex: Find the F.S. of the square wave with $p = 4$

$$f(t) = \begin{cases} -1 & -2 < t < 0 \\ +1 & 0 < t < 2 \\ 0 & t = -2, 0, 2 \end{cases}$$



If $P = 4$ then $L = 2$

First let's find a_0

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(t)dt = \frac{1}{2} \int_{-2}^{2} f(t)dt$$

$$= \frac{1}{2} \left[\int_{-2}^{0} (-1)dt + \int_{0}^{2} (+1)dt \right]$$

$$= \frac{1}{2} \left[(-t)\Big|_{-2}^{0} + (t)\Big|_{0}^{2} \right] = \frac{1}{2}\left[0 - (+(-2)) + (2-0) \right]$$

$$= \frac{1}{2}\left[(-t)_{-2}^{0} + (t)_{0}^{2} \right] = \frac{1}{2}\left[0 - (+(-2)) + (2-0) \right]$$

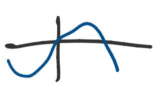
$$= \frac{1}{2}\left[-2 + 2 \right] = 0 \qquad \boxed{a_0 = 0}$$

Next calculate the a_n

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt = \frac{1}{2}\int_{-2}^{2} f(t)\cos\left(\frac{n\pi t}{2}\right)dt$$

$$= \frac{1}{2}\left[\int_{-2}^{0} (-1)\cos\left(\frac{n\pi t}{2}\right)dt + \int_{0}^{2} (+1)\cos\left(\frac{n\pi t}{2}\right)dt \right]$$

$$= \frac{1}{2}\left[\left(\frac{-\sin\left(\frac{n\pi t}{2}\right)}{\left(\frac{n\pi}{2}\right)} \right)_{-2}^{0} + \left(\frac{\sin\left(\frac{n\pi t}{2}\right)}{\left(\frac{n\pi}{2}\right)} \right)_{0}^{2} \right]$$



$$= \frac{1}{2}\left(\frac{2}{n\pi}\right)\left[-\cancel{\sin(0)}^{0} - \left(-\sin\left(\frac{-2n\pi}{2}\right)\right) + \sin\left(\frac{2n\pi}{2}\right) - \cancel{\sin(0)}^{0} \right]$$

$\sin(-x) = -\sin(x)$

$$= \frac{1}{n\pi}\left[\sin(-n\pi) + \sin(n\pi) \right] = \frac{1}{n\pi}\left[-\sin(n\pi) + \sin(n\pi) \right] = 0$$

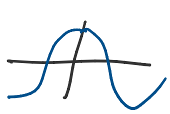
$$\boxed{a_n = 0}$$

Now calculate the b_n

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt = \frac{1}{2}\int_{-2}^{2} f(t)\sin\left(\frac{n\pi t}{2}\right)dt$$

$$= \frac{1}{2}\left[\int_{-2}^{0} (-1)\sin\left(\frac{n\pi t}{2}\right)dt + \int_{0}^{2} (+1)\sin\left(\frac{n\pi t}{2}\right)dt \right]$$

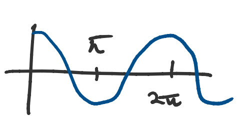
$$= \frac{1}{2}\left[\left(\frac{\cos\left(\frac{n\pi t}{2}\right)}{\frac{n\pi}{2}} \right)_{-2}^{0} + \left(\frac{-\cos\left(\frac{n\pi t}{2}\right)}{\frac{n\pi}{2}} \right)_{0}^{2} \right]$$



$$= \frac{1}{2}\left(\frac{2}{n\pi}\right)\left[\cos(0) - \cos\left(\frac{-2n\pi}{2}\right) + -\cos\left(\frac{2n\pi}{2}\right) - (-\cos(0)) \right]$$

$\cos(-x) = \cos(x)$

$$= \frac{1}{n\pi}\left[1 - \cos(n\pi) - \cos(n\pi) + 1 \right]$$



$$= \frac{1}{n\pi}\left[2 - 2\cos(n\pi) \right] = \frac{2}{n\pi}\left[1 - (-1)^n \right] = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$= \frac{1}{n\pi}\Big[2 - 2\underbrace{\cos(n\pi)}_{(-1)^n}\Big] = \frac{2}{n\pi}\left[1-(-1)^n\right] = \begin{cases} \quad n\pi \\ 0 & n \text{ even}\end{cases}$$

$\boxed{a_0 = 0}$ $\boxed{a_n = 0}$ $\boxed{b_n = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even}\end{cases}}$

Fourier series

$$f(t) \sim \cancelto{0}{\frac{a_0}{2}} + \sum_{n=1}^{\infty}\left(\cancelto{0}{a_n}\cos\left(\frac{n\pi}{2}t\right) + b_n \sin\left(\frac{n\pi t}{2}\right)\right)$$

$$= \sum_{n \text{ odd}} \frac{4}{n\pi}\sin\left(\frac{n\pi t}{2}\right)$$

$$\boxed{f(t) \sim \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\sin\left(\frac{n\pi t}{2}\right)}$$

II. Convergence:

In Lec 15, sometimes the F.S. doesn't converge to $f(t)$ at every point t.

Q: When does the Fourier series converge?

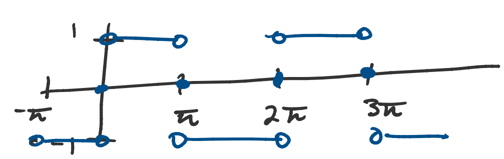
Def: A function f is piecewise continuous on $[a,b]$ provided there is a finite partition of $[a,b]$

$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$ such that

1. f is continuous on $t_i < t < t_{i+1}$
2. At each endpoint t_i, the limit of $f(t)$ as $t \to t_i$ from within the interval exists and is finite.

Ex: The square wave with period $p = 2\pi$

$$f(t) = \begin{cases} -1 & -\pi < t < 0 \\ +1 & 0 < t < \pi \\ 0 & t = -\pi, 0, \pi \end{cases}$$



Show piecewise continuous on one period, $-\pi < t < \pi$

Show piecewise continuous on one period, $-\pi < t < \pi$

partition $-\pi = t_0 < t_1 = 0 < t_2 = \pi$

1. f is continuous on each subinterval

   on $-\pi < t < 0$ $\quad f(t) = -1 \longrightarrow$ continuous ✓

   on $0 < t < \pi$ $\quad f(t) = +1 \longrightarrow$ continuous ✓

2. At each t_i, $\lim\limits_{t \to t_i^{\pm}} f(t)$ exists and is finite.

   @ $t = -\pi$ $\quad \lim\limits_{t \to -\pi^+} f(t) = \lim\limits_{t \to -\pi^-} (-1) = -1$ exists finite ✓

   @ $t = 0$ $\quad \lim\limits_{t \to 0^-} f(t) = \lim\limits_{t \to 0^-} (-1) = -1$ ✓

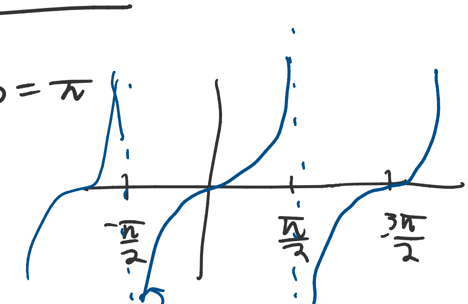
   @ $t = 0$ $\quad \lim\limits_{t \to 0^+} f(t) = \lim\limits_{t \to 0^+} (+1) = +1$ ✓

   @ $t = \pi$ $\quad \lim\limits_{t \to \pi^-} f(t) = \lim\limits_{t \to \pi^-} (+1) = +1$ ✓

So $f(t)$ is <u>piece wise continuous</u>

<u>Ex</u>: (counter example) $g(t)$ with $p = \pi$

$g(t) = \tan(t)$ on $-\frac{\pi}{2} < t < \frac{\pi}{2}$



continuous on the interval $-\frac{\pi}{2} < t < \frac{\pi}{2}$ (1) ✓

but $\lim\limits_{t \to \frac{\pi}{2}^-} g(t) = \lim\limits_{t \to \frac{\pi}{2}^-} \tan(t) = +\infty$ not finite

X condition 2

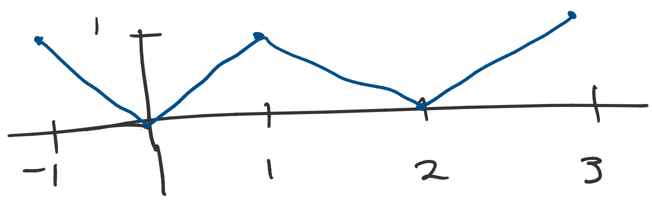
so $g(t)$ is <u>NOT</u> <u>piecewise continuous</u>

<u>Def</u>: The piecewise continuous function $f(t)$ is said to be <u>piecewise smooth</u> provided that its derivative $f'(t)$ is piecewise continuous.

be <u>piecewise smooth</u> provided

$f'(t)$ is piecewise continuous.

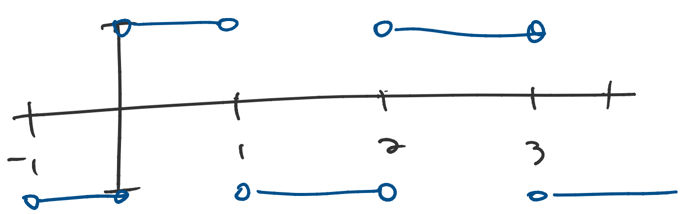
<u>Ex</u>: $f(t) = \{ |t|$ if $-1 \le t \le 1 \}$ with $p=2$



piecewise continuous function

<u>Q</u>: is it piecewise smooth?

$$f'(t) = \begin{cases} -1 & \text{if } -1 < t < 0 \\ +1 & \text{if } 0 < t < 1 \\ \text{undef} & \text{if } t = 0, -1, 1 \end{cases}$$



piecewise continuous

so yes $f(t)$ is <u>piecewise smooth</u>. ✓

<u>Thm</u>: (Convergence of Fourier Series)

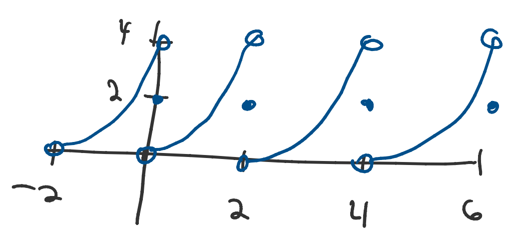
If f is a periodic function that is piecewise smooth then its Fourier series converges

(a) to the value $f(t)$ at each point where f is continuous

(b) to the value $\frac{1}{2}[f(t^+) + f(t^-)]$ at each point where f is discontinuous.

<u>Ex</u>: $f(t)$ periodic $p = 2$

$$f(t) = \begin{cases} t^2 & \text{if } 0 < t < 2 \\ 0 & \text{if } t = 0, 2 \end{cases}$$



@ $t=2$ $\quad f(2^-) = \lim_{t\to 2^-} f(t) = \lim_{t \to 2^-} t^2 = 4$

$$f(2^+) = \lim_{t\to 2^+} f(t) = 0$$

the F.S. will converge to

$$\frac{1}{2}\left[f(2^-) + f(2^+)\right] = \frac{1}{2}\left[4+0\right] = 2$$

so this function, the F.S. will converge for all t.

Find the F.S. $\rightarrow$ first find a_0

$$a_0 = \frac{1}{2}\int_0^{2L} f(t)\,dt = \frac{1}{1}\int_0^2 t^2\,dt = \left(\frac{t^3}{3}\right)_0^2 = \frac{8}{3}$$

$$\boxed{a_0 = \frac{8}{3}}$$

Find the a_n

$$a_n = \int_0^2 f(t)\cos\left(\frac{n\pi t}{1}\right)dt = \int_0^2 t^2 \cos(n\pi t)\,dt$$

$u = n\pi t \qquad t = \frac{u}{n\pi} \qquad t^2 = \left(\frac{u}{n\pi}\right)^2$

$dt = \frac{du}{n\pi}$

$$= \frac{1}{(n\pi)^3}\int_0^{2n\pi} u^2\cos(u)\,du$$

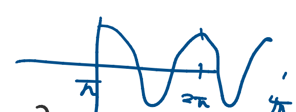
want to use

$\int u^2\cos(u)\,du = u^2\sin(u) - 2\int u\sin(u)\,du$

$$= \frac{1}{(n\pi)^3}\left[\left(u^2\sin(u)\right)_0^{2n\pi} - 2\int_0^{2n\pi} u\sin(u)\,du\right]$$

use $\int u\sin(u)\,du = \sin u - u\cos(u)$

$$= \frac{1}{(n\pi)^3}\left[u^2\sin(u) - 2\sin u + 2u\cos(u)\right]_0^{2n\pi}$$



$$= \frac{1}{(n\pi)^3}\Big[(2n\pi)^2\overset{0}{\cancel{\sin(2n\pi)}} - 0 - 2\overset{0}{\cancel{\sin(2n\pi)}} + 2\overset{0}{\cancel{\sin(0)}} + 2(2n\pi)\underbrace{\cos(2n\pi)}_{(1)} - 0\Big]$$

$$= \frac{1}{(n\pi)^3}\left[4n\pi(1)\right] = \frac{4}{n^2\pi^2}$$

$$\boxed{a_n = \frac{4}{n^2\pi^2}}$$

Calculate the bn

Calculate the b_n

use $\int u^2 \sin(u)\,du = -u^2\cos(u) + 2\int u\cos(u)\,du$

$u = n\pi t \quad t = \frac{u}{n\pi} \quad t^2 = \left(\frac{u}{n\pi}\right)^2 \quad dt = \frac{du}{n\pi}$

$$b_n = \int_0^2 t^2 \sin(n\pi t)\,dt$$

$$= \frac{1}{(n\pi)^3}\int_0^{2n\pi} u^2 \sin(u)\,du$$

use $\int u\cos(u)\,du = \cos(u) + u\sin(u)$

$$= \frac{1}{(n\pi^3)}\left[\left(-u^2\cos(u)\right)_0^{2n\pi} + 2\int_0^{2n\pi} u\cos(u)\,du\right]$$

$$= \frac{1}{(n\pi)^3}\left[-u^2\cos(u) + 2\left(\cos(u) + u\sin(u)\right)\right]_0^{2n\pi}$$

$$= \frac{1}{(n\pi)^3}\Big[-(2n\pi)^2\overset{1}{\cos(2n\pi)} - (0) + 2\overset{1}{\cos(2n\pi)} - 2\overset{1}{\cos(0)}$$
$$+ 2(2n\pi)\cancel{\sin(2n\pi)}^{0} - 0\Big]$$

$$= \frac{1}{(n\pi)^3}\left[-4n^2\pi^2(1) + \cancel{2(1)} - \cancel{2(1)}\right] = \frac{-4}{n\pi}$$

$a_0 = \frac{8}{3} \quad a_n = \frac{4}{n^2\pi^2} \quad \boxed{b_n = \frac{-4}{n\pi}}$

Putting it all together

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\pi t) + b_n\sin(n\pi t)$$

$$= \frac{1}{2}\left(\frac{8}{3}\right) + \sum_{n=1}^{\infty}\left[\left(\frac{4}{n^2\pi^2}\right)\cos(n\pi t) + \left(\frac{-4}{n\pi}\right)\sin(n\pi t)\right]$$

$$\boxed{f(t) = \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(n\pi t)}{n^2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi t)}{n}}$$

= there, because by the convergence thm, the F.S. converges for all t, even at the points of discontinuity.

Draw conclusions from the Fourier Series

Let $t=0$

$$f(0) = 2 = \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\overbrace{\cos(0)}^{=1}}{n^2} - \cancel{\frac{4}{\pi}\cdot\sum_{n=1}^{\infty}\frac{\sin(0)}{n}}^{\,0}$$

$$2 = \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Rearrange

$$\frac{\pi^2}{6} = \frac{\pi^2}{4}\cdot\frac{2}{3} = \frac{\pi^2}{4}\left(2-\frac{4}{3}\right) = \boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}$$

this series was discovered by Euler

Let $t=1$

$$f(1) = 1 = \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\overbrace{\cos(n\pi)}^{(-1)^n}}{n^2} - \cancel{\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi)}{n}}^{\,0}$$

$$1 = \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$(-1)\cdot -\frac{\pi^2}{12} = \frac{\pi^2}{4}\left(\frac{-1}{3}\right) = \frac{\pi^2}{4}\left(1-\frac{4}{3}\right) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\,(-1)$$

$$\boxed{\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}}$$

Let's take the average of these two series

$$\frac{1}{2}\left[\frac{2\cdot\pi^2}{2\cdot 6} + \frac{\pi^2}{12}\right] = \frac{1}{2}\left[\sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\right]$$

$$\frac{1}{2}\left(\frac{3\pi^2}{12}\right] = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2} + \frac{(-1)^{n+1}}{n^2}\right) \rightarrow \begin{cases} \frac{2}{n^2} & n \text{ even} \\ 0 & n \text{ even} \end{cases}$$

$$\boxed{\frac{\pi^2}{8} = \sum_{n \text{ even}}\frac{1}{n^2}}$$