Lecture 17 (Sec 9.3)

★ Fourier Sine & Cosine Series

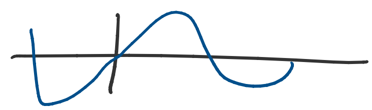
Def: A function f is said to be even if

$$f(-t) = f(t) \quad \text{for all } t$$

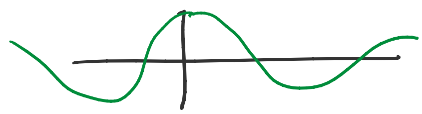
and is said to be odd if

$$f(-t) = -f(t) \quad \text{for all } t$$

Ex: $\sin(t)$ is odd



$\cos(t)$ is even



Furthermore

if f is even $\quad \int_{-a}^{a} f(t)\,dt = 2\int_0^a f(t)\,dt$

if f is odd $\quad \int_{-a}^{a} f(t)\,dt = 0$

Note:

1. If $f(t)$ is periodic even fcn, then the F.S. will only have cos terms
   $b_n = 0 \qquad a_n \neq 0$
2. If $f(t)$ is periodic odd fcn then the F.S. will only have sin terms
   $b_n \neq 0 \qquad a_n = 0$

I. Even and Odd Extensions:

Consider $f(t)$ defined $0 < t < L$

Goal: expand $f(t)$ to be a periodic function for all t, period $P = 2L$

Two choices:

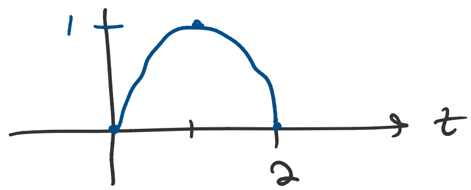
The <u>even period 2L extension</u> of f

$$f_E(t) = \begin{cases} f(t) & 0<t<L \\ f(-t) & -L<t<0 \end{cases}$$

The <u>odd period 2L extension</u> of f

$$f_O(t) = \begin{cases} f(t) & 0<t<L \\ -f(-t) & -L<t<0 \end{cases}$$

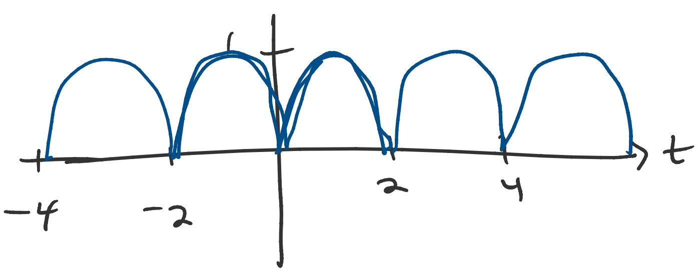
<u>Ex</u>: $f(t) = 2t - t^2$ on $0<t<2$



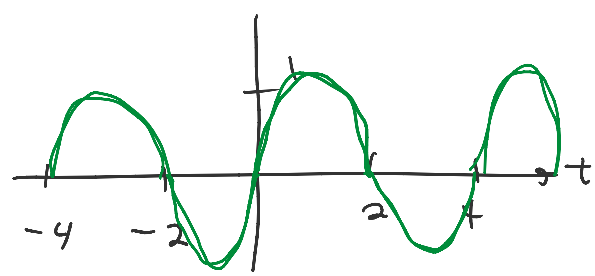
$t=0$ $f=0$
$t=1$ $f=1$
$t=2$ $f=0$

quadratic polynomial

Then even extension



The odd extension



<u>Def</u>: Suppose $f(t)$ is piecewise continuous on the interval $[0, L]$. Then the <u>Fourier cosine series</u> of f is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right)$$

with $a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi t}{L}\right)dt$

The <u>Fourier sine series</u> of f is:

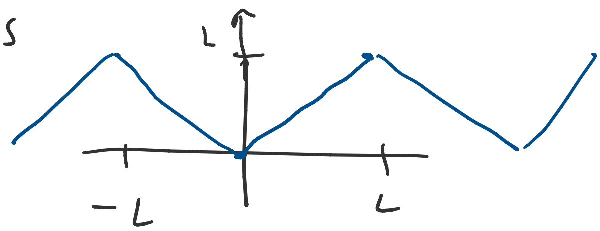
$$f(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$$

with $b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)dt$

<u>Ex</u>: $f(t) = t$ for $0 < t < L$

Find the Fourier cosine series

$$f_E(t) = \begin{cases} t & 0<t<L \\ -t & -L<t<0 \end{cases}$$



$$a_0 = \frac{2}{L}\int_0^L t\,dt = \frac{2}{L}\left(\frac{t^2}{2}\right)\Big|_0^L = \frac{2}{L}\left(\frac{L^2}{2} - 0\right) = L$$

$$a_n = \frac{2}{L}\int_0^L t\cos\left(\frac{n\pi t}{L}\right)dt$$

$\int u\cos(u)\,du = u\sin(u) + \cos(u)$

$u = \frac{n\pi t}{L}$ $\quad t = \frac{Lu}{n\pi}$ $\quad dt = \frac{L\,du}{n\pi}$

$$= \frac{2}{L}\left(\frac{L}{n\pi}\right)^2 \int_0^{n\pi} u\cos(u)\,du$$

$$= \frac{2L}{n^2\pi^2}\Big[u\sin(u) + \cos(u)\Big]_0^{n\pi}$$

$$= \frac{2L}{n^2\pi^2}\Big[(n\pi)\cancel{\sin(n\pi)}^{\,0} - 0 + \underbrace{\cos(n\pi)}_{(-1)^n} - \overset{=1}{\cos(0)}\Big]$$

$$= \frac{2L}{n^2\pi^2}\left[(-1)^n - 1\right] = \begin{cases} \frac{-4L}{n^2\pi^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

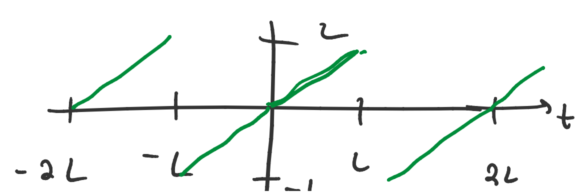
So the Fourier cosine series is

$$\frac{L}{2} + \sum_{n \text{ odd}} \left(\frac{-4L}{n^2\pi^2}\right)\cos\left(\frac{n\pi t}{L}\right)$$

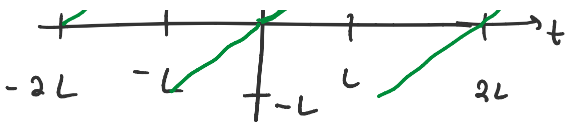
$$= \frac{L}{2} - \frac{4L}{\pi^2}\cos\left(\frac{\pi t}{L}\right) - \frac{4L}{9\pi^2}\cos\left(\frac{3\pi t}{L}\right) - \cdots$$

Find the Fourier sine series

$$f_O(t) = \begin{cases} t & 0<t<L \\ t & -L<t<0 \end{cases}$$



$$f_o(t) = \begin{cases} t & 0 < t < L \\ t & -L < t < 0 \end{cases}$$



Calculate the b_n

$$b_n = \frac{2}{L}\int_0^L t \sin\left(\frac{n\pi t}{L}\right) dt \quad \leftarrow \text{exercise}$$

$$= \frac{2L}{n\pi}(-1)^{n+1}$$

So the Fourier series

$$\sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1} \sin\left(\frac{n\pi t}{L}\right)$$

$$= \frac{2L}{\pi}\left[\sin\left(\frac{\pi t}{L}\right) - \frac{1}{2}\sin\left(\frac{2\pi t}{L}\right) + \frac{1}{3}\sin\left(\frac{3\pi t}{L}\right) - \dots\right]$$

II. Solving Differential Eqns:

First, we need to figure out how to handle derivatives

Say $f(t)$ is continuous and periodic, and its derivative $f'(t)$ is piecewise smooth

$f(t)$ has a F.S.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right)$$

We can find F.S. of $f'(t)$ by taking a termwise deriv of F.S. of $f(t)$

$$f'(t) = \frac{d}{dt} f(t) = \frac{d}{dt}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right)\right]$$

because $\frac{d}{dt}$ is linear

$$= \frac{d}{dt}\left(\frac{a_0}{2}\right) + \frac{d}{dt}\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)$$

$$= \frac{d}{dt}\left(\frac{a_0}{2}\right) + \frac{d}{dt}\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \ldots$$

because $\frac{d}{dt}$ is linear

$$= \cancel{\frac{d}{dt}\left(\frac{a_0}{2}\right)}^{\,0} + \sum_{n=1}^{\infty} \frac{d}{dt}\left(a_n \cos\left(\frac{n\pi t}{L}\right)\right) + \sum_{n=1}^{\infty} \frac{d}{dt}\left(b_n \sin\left(\frac{n\pi t}{L}\right)\right)$$

$$= \sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{L}\right)\left(-\sin\left(\frac{n\pi t}{L}\right)\right) + \sum_{n=1}^{\infty} b_n \left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi t}{L}\right)$$

$$\boxed{f'(t) = \sum_{n=1}^{\infty} -\frac{n\pi a_n}{L} \sin\left(\frac{n\pi t}{L}\right) + \frac{n\pi b_n}{L} \cos\left(\frac{n\pi t}{L}\right)}$$

Def: An endpoint value problem is ODE with the following conditions

$$a x'' + b x' + c x = f(t) \qquad (0 < t < L)$$

$$x(0) = \underbrace{x(L)}_{x \text{ evaluated at } t = L} = 0$$

(an IVP, we would have $x'(0) = x_0'$)

To solve this using F.S.

1. extend $f(t)$ to be a periodic fcn.
2. Then find the F.S. of $f(t)$

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi t}{L}\right) + B_n \sin\left(\frac{n\pi t}{L}\right)$$

known

3. Assume $x(t)$ has a F.S.

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)$$

unknown

unknown
need to find

4. Plug $x(t)$ into the ODE and solve for $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$

<u>Ex</u>: Find a Fourier Series solution of the endpoint value problem:

$$x'' + 4x = 4t \qquad 0<t<1$$
$$x(0) = x(1) = 0$$

Here $f(t) = 4t$ for $0<t<1$

1. Need to extend $f(t)$ as periodic fcn
odd or even extension?

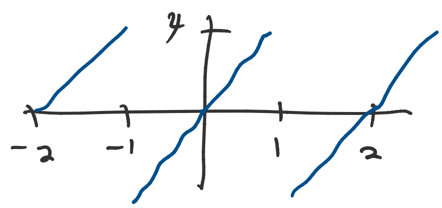
Choose $f(t)$ extension so that the F.S. satisfies $x(0)=0$ and $x(1)=0$

If f is odd, $\rightarrow$ Fourier sine series

$$\sin\left(\frac{n\pi t}{2}\right) \rightarrow \begin{matrix} t=0 & \sin(0)=0 \\ t=1 & \sin(n\pi)=0 \end{matrix}$$

Choose the <u>odd</u> extension

$$f_o(t) = \begin{cases} 4t & 0<t<1 \\ 4t & -1<t<0 \end{cases}$$



2. Find the F.S. of $f_o(t)$

$a_n = 0$

$$b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)dt \qquad \begin{matrix} L=1 \\ f(t)=4t \end{matrix}$$

$$= 2\int_0^1 4t\sin(n\pi t)\,dt$$

$\int u\sin(u)du = -u\cos(u) + \sin(u)$
$u = n\pi t \quad t = \frac{u}{n\pi} \quad dt = \frac{du}{n\pi}$

$$= \frac{2\cdot 4}{\ \ ^2}\int^{n\pi} u\sin(u)\,du$$

$$= \frac{2 \cdot 4}{(n\pi)^2} \int_0^{n\pi} u \sin(u)\, du$$

$$= \frac{8}{(n^2\pi^2)} \Big[-u\cos(u) + \sin(u) \Big]_0^{n\pi}$$

$$= \frac{8}{n^2\pi^2} \Big[-(n\pi)\underbrace{\cos(n\pi)}_{(-1)^n} - 0 + \cancel{\sin(n\pi)}^{\,0} - \cancel{\sin(0)}^{\,0} \Big]$$

$$= \frac{8}{n^2\pi^2} (n\pi)(-1)^{n+1} = \frac{8(-1)^{n+1}}{n\pi}$$

So we have

$$4t = \sum_{n=1}^{\infty} \frac{8(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

3. Assume $x(t)$ has a F.S. since $x(0) = x(1) = 0$ $\rightarrow$ Fourier sine series

$$x(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t)$$

Goal: find b_n

4. Plug $x(t)$ into the ODE $\quad x'' + 4x = 4t$

$$\frac{d^2}{dt^2}\left(\sum_{n=1}^{\infty} b_n \sin(n\pi t) \right) + 4\left(\sum_{n=1}^{\infty} b_n \sin(n\pi t) \right) = \sum_{n=1}^{\infty} \frac{8(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

since $\frac{d}{dt}$ is linear

$$\left(\sum_{n=1}^{\infty} b_n(-n^2\pi^2)\sin(n\pi t) \right) + \left(\sum_{n=1}^{\infty} 4b_n \sin(n\pi t) \right) = \sum_{n=1}^{\infty} \frac{8(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

Since all these terms are linear combine like terms inside the sum

$$\sum_{n=1}^{\infty} \underbrace{\left[-n^2\pi^2 b_n + 4b_n - \frac{8(-1)^{n+1}}{n\pi} \right]}_{\text{all of these}} \sin(n\pi t) = 0$$

in order for this to be true

all of these coefficients need to sum to zero

in order for this to be true

$$-n^2\pi^2 b_n + 4 b_n - \frac{8(-1)^{n+1}}{n\pi} = 0 \qquad \text{for all } b_n$$

$$(4 - n^2\pi^2)\, b_n = \frac{8(-1)^{n+1}}{n\pi}$$

$$b_n = \frac{8(-1)^{n+1}}{n\pi(4 - n^2\pi^2)}$$

So now, we can write

$$x(t) = \sum_{n=1}^{\infty} \frac{8(-1)^{n+1}}{n\pi(4 - n^2\pi^2)} \sin(n\pi t)$$

$$\boxed{x(t) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(4 - n^2\pi^2)} \sin(n\pi t)}$$

<u>Note</u>: This is very powerful concept numerical solvers for ODEs / PDEs

<u>Spectral Methods</u>

Needs endpoint condition $x(0) = x(1) = 0$