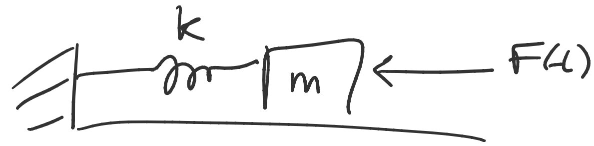
☆ Applications of Fourier Series

I. Forced Oscillations:



$$mx'' + kx = F(t)$$

$F(t)$ - periodic external force

$w_0 = \sqrt{\frac{k}{m}}$ natural frequency

general solution:

$$x(t) = \underbrace{C_1 \cos(w_0 t) + C_2 \sin(w_0 t)}_{\text{homogeneous}} + \underbrace{x_p(t)}_{\text{particular}}$$

Want: Use Fourier series to find a periodic particular solution. Call this the steady periodic solution

$x_{sp}(t)$ → periodic
→ particular

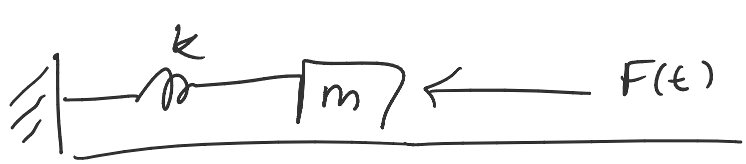
Assume $F(t)$ is an odd fcn with $p = 2L$

$$F(t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi t}{L}\right)$$

⊛ under most conditions, we can find a soln

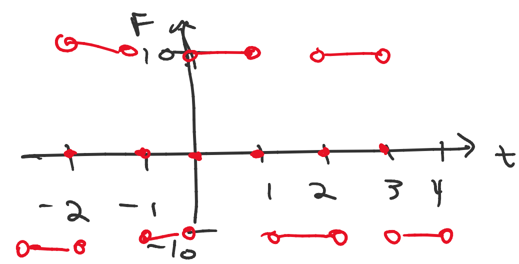
$$x_{sp}(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$$

Ex:



$m = 2kg$
$k = 32N/m$
$F(t)$ - odd $P = 2s$

$F(t)$ — odd $T = 2$



$$2x'' + 32x = F(t)$$

$$F(t) = \begin{cases} +10N & \text{if } 0 < t < 1 \\ -10N & \text{if } 1 < t < 2 \end{cases}$$

The natural freq $\omega_0 = \sqrt{\frac{32}{2}} = \sqrt{16} = 4$

homogeneous soln

$$x_h(t) = C_1 \cos(4t) + C_2 \sin(4t)$$

The general soln

$$x(t) = x_h(t) + x_{sp}(t)$$

Find $x_{sp}(t)$.

1. Find the F.S. of $F(t)$ — odd $\rightarrow$ Fourier sine series

$A_0 = 0 \quad A_n = 0$

$$B_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{2}\right)dt = 2\int_0^1 10\sin(n\pi t)\,dt$$

$$= 20\left[-\frac{\cos(n\pi t)}{n\pi}\right]_0^1 = \frac{20}{\pi}\Big[\underbrace{-\cos(n\pi)}_{(-1)^{n+1}} + \cancel{\cos}^{\,1}(0)\Big]$$

$$= \frac{20}{\pi}\left[(-1)^{n+1} + 1\right] = \begin{cases} \frac{40}{n\pi} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

so $$F(t) = \frac{40}{\pi}\sum_{n \text{ odd}} \frac{\sin(n\pi t)}{n}$$

2. Assume $$x_{sp}(t) = \sum_{n \text{ odd}} b_n \sin(n\pi t)$$

Since $F(t)$ is odd,

Take derivatives and plug into ODE

Since $F(t)$ is odd, $x_{sp}(t)$ is odd

$a_n = 0$

also sum over n odd

Take derivatives and plug into ODE

$$x_{sp}''(t) = \sum_{n \text{ odd}} b_n(-n^2\pi^2)\sin(n\pi t)$$

$$2x'' + 32x = F(t)$$

$$2\left(\sum_{n \text{ odd}} -b_n n^2\pi^2 \sin(n\pi t)\right) + 32\left(\sum_{n \text{ odd}} b_n \sin(n\pi t)\right) = \left(\frac{40}{\pi}\sum_{n \text{ odd}} \frac{\sin(n\pi t)}{n}\right)$$

Rearrange:

$$\sum_{n \text{ odd}} \underbrace{\left[-2b_n n^2\pi^2 + 32b_n - \frac{40}{\pi n}\right]}_{\text{must be equal to zero}} \sin(n\pi t) = 0$$

$$-2n^2\pi^2 b_n + 32b_n - \frac{40}{\pi n} = 0$$

$$(32 - 2n^2\pi^2)b_n = \frac{40}{\pi n}$$

$$b_n = \frac{40}{\pi n(32 - 2n^2\pi^2)}$$

$$b_n = \frac{20}{\pi n(16 - n^2\pi^2)}$$

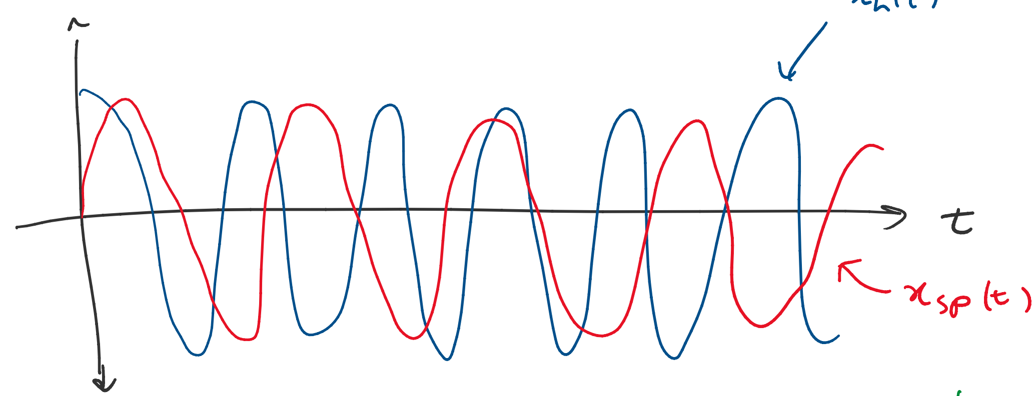
$$\boxed{x_{sp}(t) = \frac{20}{\pi}\sum_{n \text{ odd}} \frac{\sin(n\pi t)}{n(16 - n^2\pi^2)}}$$

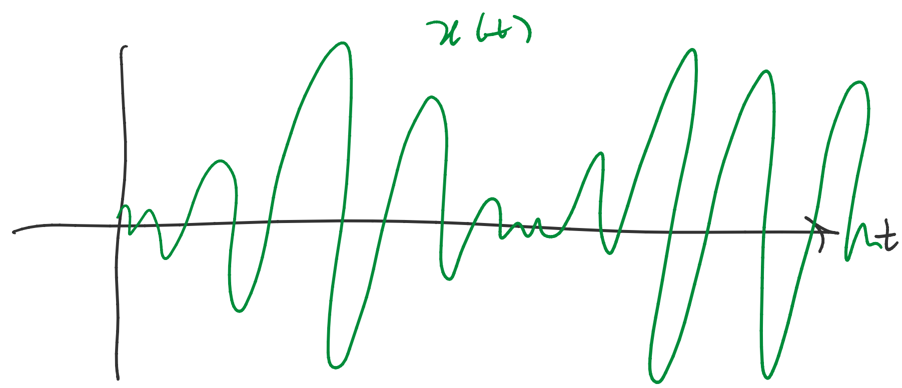
So the general soln:

$$x(t) = \underline{C_1\cos(4t) + C_2\sin(4t)} + \underline{\frac{20}{\pi}\sum_{n \text{ odd}} \frac{\sin(n\pi t)}{n(16-n^2\pi^2)}}$$

$$x(t) = \underbrace{C_1\cos(4t) + C_2\sin(4t)}_{x_h(t) \text{ has a period } \frac{2\pi}{4} = \frac{\pi}{2}} + \underbrace{\frac{[illegible]}{\pi}\sum_{n \text{ odd}} \frac{[illegible]}{n(16-n^2\pi^2)}}_{\text{has a period } 2}$$

The frequencies will interact, causing "beats"





because we are add two periodic fcns with slightly different frequencies

II. Pure Resonance:

(*) Caveat:

When $F(t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi t}{L}\right)$

$mx'' + kx = F(t)$ $\qquad w_0 = \sqrt{\frac{k}{m}}$

If there is N such that $\boxed{\frac{N\pi}{L} = w_0}$

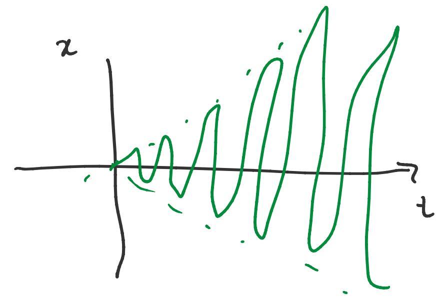
This term causes pure resonance

Recall: $mx'' + kx = B_N \sin(\omega_0 t)$

has resonance solution

$$x(t) = -\frac{B_N}{2m\omega_0}\, t \cos(\omega_0 t)$$

factor of t!



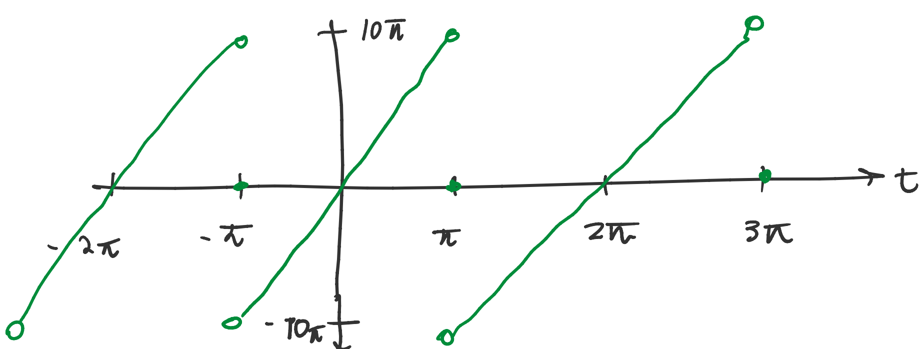
Then full solution

$$x_{sp}(t) = \underbrace{\frac{-B_N}{2m\omega_0}\, t\cos(\omega_0 t)}_{\text{this term blows up as } t \to \infty} + \sum_{n \neq N} \frac{B_n}{n\left(\omega_0^2 - \frac{n^2\pi^2}{L^2}\right)} \sin\left(\frac{n\pi t}{L}\right)$$

Ex: Suppose $m = 2\text{kg}$ $k = 32$ N/m $\omega_0 = 4$

$$2x'' + 32k = F(t)$$

$$F(t) = \{\ 10t \quad \text{on} \quad -\pi < t < \pi$$



$P = 2\pi$

$L = \pi$

odd fcn

Fourier Sine Series

Here: $F(t) = 20 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nt)$

$\omega_0 = 4 = n \rightarrow$ when $n = 4$ we have pure resonance

$$\boxed{x_{sp}(t) = \frac{5}{8}\, t\cos(4t) + 10 \sum_{n \neq 4} \frac{(-1)^{n+1}}{n(16 - n^2)} \sin(nt)}$$

III. Damped Forced Oscillations

Consider mass-spring-dashpot system:

$$mx'' + cx' + kx = F(t)$$

Recall, that the solution when $F(t) = F_0 \sin(\omega t)$ is:

$$x(t) = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}} \sin(\omega t - \alpha)$$

where $\alpha = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$

If $F(t)$ is an odd periodic fcn w/ F.S.

$$F(t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi t}{L}\right)$$

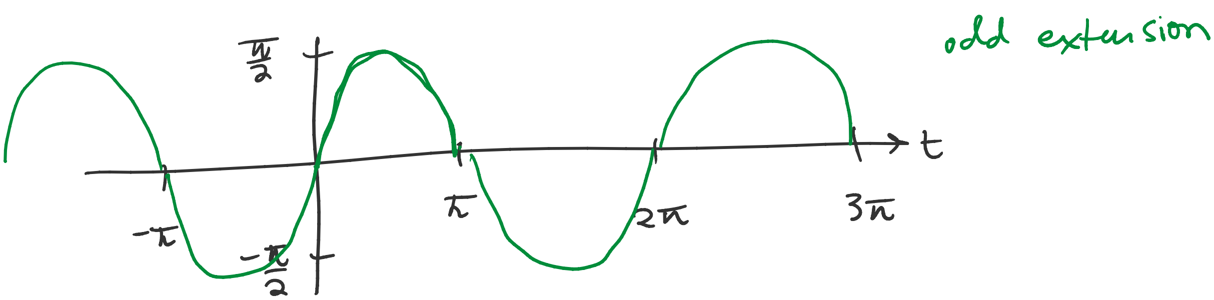
then by supperposition

$$\boxed{x_{sp}(t) = \sum_{n=1}^{\infty} \frac{B_n \sin(\omega_n t - \alpha_n)}{\sqrt{(k - m\omega_n^2)^2 + (c\omega_n)^2}} \quad \text{where } \omega_n = \frac{n\pi}{L} \text{ and } \alpha_n = \tan^{-1}\left(\frac{c\omega_n}{k - m\omega_n^2}\right)}$$

Ex: $m = 3\,kg$ $\quad c = 0.02\, N/m/s$ $\quad k = 27\, N/m$

$$3x'' + 0.02x' + 27x = F(t)$$

$$F(t) = \{\pi t - t^2 \text{ if } 0 < t < \pi\}$$



F(t) is:

The F.S. of $F(t)$ is:

$$F(t) = \frac{8}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n^3} \qquad B_n = \begin{cases} \frac{8}{\pi n^3} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$w_n = n$$

$$x_{sp}(t) = \sum_{n \text{ odd}} \frac{\left(\frac{8}{\pi n^3}\right) \sin(nt - \alpha_n)}{\sqrt{(27 - 3n^2)^2 + (0.02n)^2}}$$

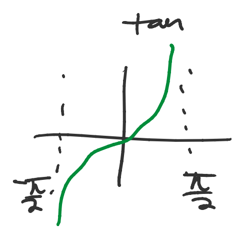
with $\alpha_n = \tan^{-1}\left(\frac{0.02n}{27 - 3n^2}\right) \qquad 0 \le \alpha_n \le \pi$

Using a calculator, let's find the first few terms

when $n = 1$

$$\alpha_1 = \tan^{-1}\left(\frac{0.02 \cdot (1)}{27 - 3(1)^2}\right) = \tan^{-1}\left(\frac{0.02}{24}\right) = 0.0008$$

$$b_1 = \frac{\left(\frac{8}{\pi (1)^3}\right)}{\sqrt{(27 - 3 \cdot (1)^2)^2 + (.02(1))^2}} = 0.1061$$



when $n = 3$

$$\alpha_3 = \tan^{-1}\left(\frac{0.02(3)}{27 - 3(3)^2}\right) = \frac{\pi}{2}$$

$$b_3 = \frac{\left(\frac{8}{\pi (3)^3}\right)}{\sqrt{(27 - 3 \cdot (3)^2)^2 + (0.02(3))^2}} = \frac{8}{27\pi(0.02 \cdot 3)} = 1.5719$$

$$
\boxed{\begin{aligned} x_{sp} \approx\ & (0.1061)\sin(t - 0.0008) \\ & + (1.5719)\sin\left(3t - \tfrac{\pi}{2}\right) + \\ & + (0.0004)\sin(5t - 3.1437) + \cdots \end{aligned}}
$$