Lecture 19 (Sec 9.5 - Part 1)

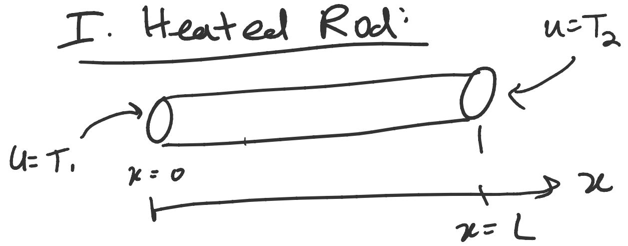
☆ Heat Conduction & Separation of Variables

Def: A <u>partial differential equation</u> (PDE) is one containing one or more partial derivatives of a fcn of multiple variables

Ex:

- $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \iff u_t = u_x$
- $u_t = u u_x \qquad (u(t,x))$
- $u_t = u_{xx} + u_{yy} \qquad (u(t,x,y))$

I. Heated Rod:



$u(x,t)$ — temperature of rod

1D - Heat Equation:

$$u_t = k u_{xx} \quad \text{on } 0 \le x \le L, \; t > 0$$

Here k is called <u>thermal diffusivity</u> $(k>0)$

Boundary Conditions (BC)

$$\begin{cases} u(0,t) = T_1 & \text{left side temp fixed at } T_1 \\ u(L,t) = T_2 & \text{right side fixed} \\ u(x,0) = f(x) & \text{initial temp @ } t=0 \end{cases}$$

Put it all together:

$$(*) \begin{cases} u_t = k u_{xx} \quad \text{on } 0 \le x \le L, \; t > 0 \\ u(0,t) = T_1 \\ u(L,t) = T_2 \\ u(x,0) = f(x) \end{cases}$$

<u>Boundary Value Problem</u> (BVP)

Def: If $u(0,t) = u(L,t) = 0$ we call it <u>homogeneous BC</u>

Note: The heat equation $u_t = k u_{xx}$ is linear, so the principle of superposition still holds:

if $u_1, u_2, u_3, \dots$ solve $(*)$ with homogeneous BC

$\sum^{\infty}$ ... also solves $(*)$

if $u_1, u_2, u_3, \dots$ solve (*) with homogeneous ...

then $u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$ also solves (*)

provided

1. the series converges
2. the series satisfies the initial condition
3. $u(x,t)$ is continuous.

II Method of Separation of Variables:

To solve: $\begin{cases} u_t = k u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$

Key step: Assume that $u(x,t)$ can be separated into 2 parts

$$u(x,t) = \underbrace{X(x)}_{\text{fcn of only } x} \underbrace{T(t)}_{\text{fcn of only } t}$$

Intuition: what happens in x doesn't directly affect what happens in t

Let's plug into PDE

$$u_t = \frac{\partial}{\partial t}\left(X(x)T(t)\right) = X\left[\frac{dT}{dt}\right] = XT'$$

$$u_x = \frac{\partial}{\partial x}\left(X(x)T(t)\right) = \left[\frac{dX}{dx}\right]T = X'T$$

$$u_{xx} = \frac{\partial^2}{\partial x^2}(XT) = \frac{\partial}{\partial x}(X'T) = X''T$$

so now $u_t = k u_{xx}$

$$XT' = kX''T$$

put all X terms on one side and t terms on other

$$\underbrace{\frac{X''}{X} = \frac{T'}{kT}}$$

This hold for all t and x

key step: $\dfrac{X''}{X} = f(x)$ and $\dfrac{T'}{kT} = g(t)$

if $f(x) = g(t)$ for all x and all t

only if $f(x) = g(t) = \text{constant}$

Write $\dfrac{X''}{X} = \dfrac{T'}{kT} = -\lambda$

choose $-\lambda$ for convenience

λ is called the separation constant

$\dfrac{X''}{X} = -\lambda$ $\qquad\qquad$ $\dfrac{T'}{kT} = -\lambda$

$\boxed{X'' + \lambda X = 0}$ $\qquad\qquad$ $\boxed{T' + k\lambda T = 0}$

Before we solve these ODEs, lets figure out the BC

$u(0,t) = 0 = X(0)T(t) \Rightarrow X(0) = 0$

$u(L,t) = 0 = X(L)T(t) \Rightarrow X(L) = 0$

$u(x,0) = f(x) = X(x)T(0) \Rightarrow T(0) \neq 0$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0 \end{cases} \qquad\qquad \begin{cases} T' + k\lambda T = 0 \\ T(0) \neq 0 \end{cases}$$

endpoint problem

Let's solve the X equation first

$X'' + \lambda X = 0 \qquad X(0) = X(L) = 0 \qquad \lambda > 0$

$X(x) = C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x)$

plug in BC

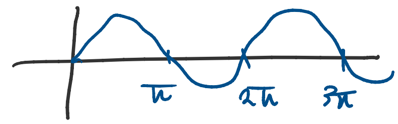
$X(0) = C_1 \cancelto{1}{\cos(0)} + C_2 \cancelto{0}{\sin(0)} = 0 \Rightarrow C_1 = 0$

$X(L) = C_2 \sin(\sqrt{\lambda}\, L) = 0$

$X(0) = C_1 \cos(0) + C_2 \sin(0)$

$X(L) = C_2 \sin(\sqrt{\lambda}\, L) = 0$

$\sin(\sqrt{\lambda}\, L) = 0$



$\sqrt{\lambda}\, L = n\pi \qquad n = 1, 2, 3, \dots$

$\lambda = \left(\frac{n\pi}{L}\right)^2 = \frac{n^2\pi^2}{L^2}$

family of solns:

$$\boxed{X_n = C_2 \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \dots}$$

Note: $X_n'' = -\lambda_n X_n \qquad \lambda_n = \frac{n^2\pi^2}{L^2}$ is an eigenvalue

and X_n is an eigenfunction

Now let's solve for T

$T' + k\lambda T = 0 \qquad \lambda_n = \frac{n^2\pi^2}{L^2}$

$T' = -\frac{kn^2\pi^2}{L^2} T \quad \leftarrow$ exponential fcn

$$\boxed{T_n = \exp\left(-kn^2\pi^2 t / L^2\right) \qquad n = 1, 2, 3, \dots}$$

put it all together

$u(x,t) = XT$

$u_n(x,t) = X_n T_n = e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \dots$

general soln $\rightarrow$ sum overs all n $\rightarrow$ principle of superposition

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)}$$

Needs to satisfy initial condition

$u(x,0) = f(x)$

Fourier sine

Needs to satisfy initial condition

$$u(x,0) = f(x)$$

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \quad \leftarrow \text{Fourier sine series of } f(x)$$

so:
$$\boxed{b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

So our final solution is:

$$\boxed{\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right) \\ b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{aligned}}$$

Ex:
$$\begin{cases} u_t = 3u_{xx} & \text{on } 0 < x < \pi,\ t > 0 \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = 4\sin(2x) \end{cases}$$

Here $k = 3 \qquad L = \pi$

$$f(x) = 4\sin(2x)$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} b_n e^{-3n^2\cancel{\pi^2} t/\cancel{\pi^2}} \sin\left(\frac{n\cancel{\pi} x}{\cancel{\pi}}\right)$$

$$= \sum_{n=1}^{\infty} b_n e^{-3n^2 t} \sin(nx)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= 2 \int^{\pi} 4\sin(2x)\sin(nx)\,dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \underbrace{4 \sin(2x) \sin(nx)\, dx}$$

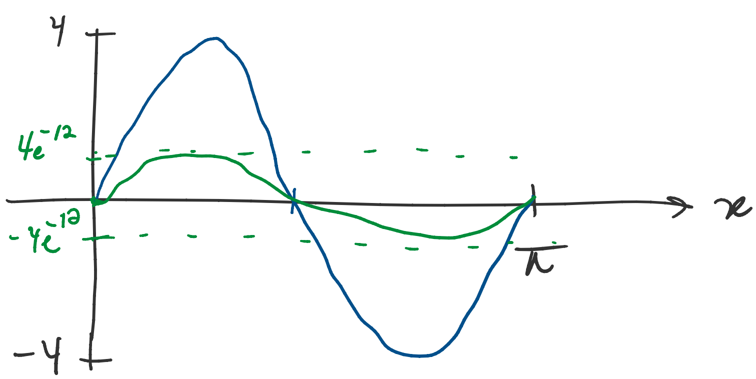
use orthogonality of $\sin(nx)$

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} \pi & n=m \\ 0 & n \neq m \end{cases}$$

$$= \frac{2 \cdot 4}{\pi} \left\{ \begin{matrix} \frac{\pi}{2} & n=2 \\ 0 & n \neq 2 \end{matrix} \right\} = \begin{cases} 4 & \text{if } n=2 \\ 0 & \text{if } n \neq 2 \end{cases}$$

$$u(x,t) = b_2 e^{-3(2)^2 t} \sin(2x)$$

$$\boxed{u(x,t) = 4e^{-12t} \sin(2x)}$$



@ $t=0$ $u(x,0) = 4\sin(2x)$

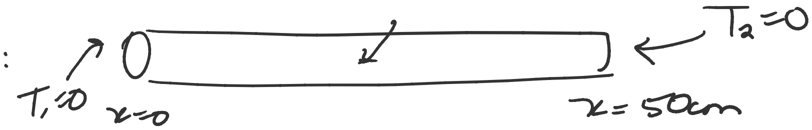
@ $t=1$ $u = 4e^{-12}\sin(2x)$

as $t \to \infty$

$e^{-12t} \to 0$

$u(x,t) \to 0$

Ex:



$f(x) = u_0$ $k = 0.15$

Rod of length $L = 50cm$, is imersed in steam until its temperature $u_0 = 100°C$.

At time $t=0$, its two ends are immersed in ice at $0°C$.

Calculate the rod's temperature at its midpoint after $t = 30min$ if its made of iron.

Eqns: $\begin{cases} u_t = k u_{xx} = 0.15 u_{xx} \\ u(0,t) = u(50,t) = 0 \end{cases}$ ← homogeneous BC

Eqns :
$$\begin{cases} u_t = k\, u_{xx} = 0.15\, u_{xx} \\ u(0,t) = u(50,t) = 0 \\ u(x,0) = u_0 = 100 \end{cases}$$
← homogeneous BC

$L = 50, \quad k = 0.15$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

where
$$b_n = \frac{2}{L}\int_0^L u_0 \sin\left(\frac{n\pi x}{L}\right) dx$$
$$= \frac{2u_0}{L}\left[\frac{-\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}\right]_0^L$$
$$= \frac{2u_0}{\cancel{L}}\left(\frac{\cancel{L}}{n\pi}\right)\left[-\cos\left(\frac{n\pi \cancel{L}}{\cancel{L}}\right) + \cos(0)\right]$$
$$= \frac{2u_0}{n\pi}\left[-(-1)^n + 1\right] = \begin{cases} \frac{4u_0}{n\pi} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

$$u(x,t) = \sum_{n \text{ odd}} \frac{4u_0}{n\pi} e^{-kn^2\pi^2 t/L} \sin\left(\frac{n\pi x}{L}\right)$$

where $k = 0.15$, $u_0 = 100°C$, $L = 50cm$

Find the temp at $t = 30$ min $= 1800s$
at $x = \frac{L}{2} = 25$ cm

Write out first few terms

$$u(25, 1800) \approx \frac{4(100)}{\pi}\left[\frac{1}{1}e^{-0.15(1)^2\pi^2(1800)/(50)^2} \sin\left(\frac{(1)\pi}{2}\right) + \frac{1}{3}e^{-0.15(3)^2\pi^2(1800)/(50)^2} \sin\left(\frac{3\pi}{2}\right) + \frac{1}{5}e^{-0.15(5)^2\pi^2(1800)/(50)^2} \sin\left(\frac{5\pi}{2}\right) + \dots\right]$$

$\approx 43.8519 - 0.0029 + 3.4e^{-10} + \cdots$

$\approx \boxed{43.85^\circ C}$ temp at $t = 30\text{min}$ at $x = \frac{L}{2}$