Lecture 1 (Sec 5.1)

☆ Matrices and Linear Systems

A system of n first-order linear differential eqns. can be written:

$$\underline{x}' = \underline{\underline{P}}(t)\,\underline{x} + \underline{f}(t)$$

$\underline{x}$: $n \times 1$ vector of variables $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$\underline{\underline{P}}(t)$: $n \times n$ matrix

$\underline{f}(t)$: $n \times 1$ vector called forcing term

When $\underline{f}(t) = \underline{0}$ the eqn is called homogeneous

Example:

$$x_1' = 4x_1 - 3x_2 + t^2$$
$$x_2' = 6x_1 - 7x_2 - e^t$$

Turns into the system

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} t^2 \\ -e^t \end{bmatrix}$$

$$\underline{x}' = \underline{\underline{P}}\,\underline{x} + \underline{f}(t)$$

A <u>solution</u> of a system of eqns is a function that satisfies the differential eqn.

Ex: Verify that $\underline{x}_1 = e^{2t}\begin{bmatrix}3\\2\end{bmatrix}$, $\underline{x}_2 = e^{-5t}\begin{bmatrix}1\\3\end{bmatrix}$

are solutions to

$$\underline{x}' = \begin{bmatrix}4 & -3\\6 & -7\end{bmatrix}\underline{x}$$

(a) $\underline{x}_1' = 2e^{2t}\begin{bmatrix}3\\2\end{bmatrix}$

so verify $\underline{x}_1' = \begin{bmatrix}4 & -3\\6 & 7\end{bmatrix}\underline{x}_1$

$$2e^{2t}\begin{bmatrix}3\\2\end{bmatrix} \stackrel{?}{=} \begin{bmatrix}4 & -3\\6 & 7\end{bmatrix} e^{2t}\begin{bmatrix}3\\2\end{bmatrix}$$

$$e^{2t}\begin{bmatrix}6\\4\end{bmatrix} \stackrel{?}{=} e^{2t}\begin{bmatrix}4\cdot3 - 3\cdot2\\6\cdot3 + 7\cdot2\end{bmatrix}$$

$$\stackrel{?}{=} e^{2t}\begin{bmatrix}6\\4\end{bmatrix}$$

Yes $\underline{x}_1$ is a solution

Exercise: show that $\underline{x}_2 = e^{-5t}\begin{bmatrix}1\\3\end{bmatrix}$

is also a solution

$\underline{x}_1$ and $\underline{x}_2$ are fundamental solutions for this system.

Note: A system of n eqns has n fundamental solns.

All linear combinations are also solutions

$$\underline{x}(t) = c_1 \underline{x}_1(t) + c_2 \underline{x}_2(t)$$

$$= c_1 e^{2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

where c_1 and c_2 are constants that relate to the initial conditions.

Principle of Superposition

The fundamental solutions must be linearly independent

Use the Wronskian to check linear independence

$$\underline{x}_1 = e^{2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} \qquad \underline{x}_2 = e^{-5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$

$$W = \begin{vmatrix} 3e^{2t} & e^{-5t} \\ 2e^{2t} & 3e^{-5t} \end{vmatrix} = 3e^{2t} \cdot 3e^{-5t} - 2e^{2t} e^{-5t}$$

$$\begin{array}{cc} \underline{x}_1 & \underline{x}_2 \end{array}$$

$$= 9e^{-3t} - 2e^{-3t}$$

$$= 7e^{-3t}$$

$W \neq 0$ so $\underline{x}_1$ and $\underline{x}_2$ are linearly independent

Initial Value Problem: (IVP)

$$\underline{x}' = \underline{P}(t)\underline{x} \quad \text{and} \quad \underbrace{\underline{x}(a) = \underline{b}}_{\text{initial condition}}$$

Example:

$$\underline{x}' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \underline{x} \quad \text{and} \quad \underline{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Recall the fundamental solutions were:

$$\underline{x}_1 = e^{2t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{and} \quad \underline{x}_2 = e^{-5t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The general solution is

$$\underline{x}(t) = c_1 e^{2t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{-5t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Goal: Find c_1 and c_2 so that $\underline{x}(t)$ satisfies the initial condition

$$\underline{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$c_1 \cancel{e^{2\cdot 0}}^{1}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 \cancel{e^{-5\cdot 0}}^{1}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$c_1\begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

By inspection $\rightarrow$ $c_1 = 1 \quad c_2 = -1$

Formally, we can write:

$$c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3c_1 + c_2 \\ 2c_1 + 3c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$3c_1 + c_2 = 2 \qquad 2c_1 + 3c_2 = -1$$

First solve in terms of c_2

$$c_2 = 2 - 3c_1$$

Now plug into 2nd eqn and solve for c_1

$$2c_1 + 3(2 - 3c_1) = -1$$

$$2c_1 + 6 - 9c_1 = 6 - 7c_1 = -1$$

$$7 = 7c_1$$

$$\boxed{c_1 = 1}$$

Plug back into eqn for c_2

$$c_2 = 2 - 3c_1 = 2 - 3 \cdot 1 = -1$$

$$\boxed{c_2 = -1}$$

So the solution to the IVP is

$$\underline{x}(t) = e^{2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} - e^{-5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$