Lecture 20 (Sec 9.5 - Part 2)

☆ Heat Conduction & Separation of Variables

Last lecture : BVP

$$\begin{cases} u_t = k u_{xx} & \text{on } 0 \le x \le L, \quad t > 0 \\ u(0,t) = u(L,t) = 0 & \text{— homogeneous BC} \\ u(x,0) = f(x) \end{cases}$$

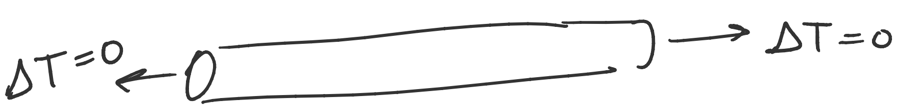
The solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

where $b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

Today — different BC

I. Insulated Endpoint Conditions



insulated endpoints
heat doesn't leave
out the ends

BC. $u_x(0,t) = u_x(L,t) = 0$

deriv wrt x

BVP: $$\begin{cases} u_t = k u_{xx} & 0 < x < L, \quad t > 0 \\ u_x(0,t) = u_x(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

To Solve → Separation of variables

$$u(x,t) = X(x)\,T(t)$$

same steps as before
up until we separate into two ODES

$$\frac{X''}{\;} = \frac{T'}{\;} = -\lambda$$

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda$$

$$X'' + \lambda X = 0 \qquad\qquad T' + \lambda k T = 0$$

new BC $\begin{cases} X'(0) = 0 \\ X'(L) = 0 \end{cases}$

$$X = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$

$$X' = -\sqrt{\lambda}\, c_1 \sin(\sqrt{\lambda}\, x) + \sqrt{\lambda}\, c_2 \cos(\sqrt{\lambda}\, x)$$

$$X'(0) = 0 = \sqrt{\lambda}\, c_2 \cos(0) \implies c_2 = 0$$

$$X'(L) = 0 = -\sqrt{\lambda}\, c_1 \sin(\sqrt{\lambda}\, L)$$

$$\sin(\sqrt{\lambda}\, L) = 0$$

$$\sqrt{\lambda}\, L = n\pi \qquad n = 0, 1, 2, 3, \dots$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2 = \frac{n^2\pi^2}{L^2} \qquad \underline{\text{eigenvalue}}$$

$$X_n = \cos\left(\frac{n\pi x}{L}\right) \qquad \underline{\text{eigenfunction}}$$

$$T' + k\lambda T = 0 \qquad \lambda = \frac{n^2\pi^2}{L^2}$$

same soln: $T_n = e^{-kn^2\pi^2 t/L^2}$ $\qquad n = 0, 1, 2, 3, \dots$

fundamental solns

$$u_n = X_n T_n = e^{-kn^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right)$$

when $n = 0$, $\quad u_0 = e^0 \cdot \cos(0) = 1$

So the full solution

$$\boxed{\begin{aligned} u(x,t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-kn^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right) \\ \text{where} \quad a_0 &= \frac{2}{L}\int_0^L f(x)\,dx \\ a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \end{aligned}}$$

<u>Ex:</u> BVP
$$\begin{cases} 3u_t = u_{xx} & 0<x<2, \quad t>0 \\ u_x(0,t) = u_x(2,t) = 0 \\ u(x,0) = \cos^2(2\pi x) \end{cases}$$

Here $L=2$, $k=\frac{1}{3}$, $f(x) = \cos^2(2\pi x)$

Solution has the form

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\frac{1}{3}n^2\pi^2 t/4} \cos\left(\frac{n\pi x}{2}\right)$$

where $a_0 = \frac{2}{2}\int_0^2 \cos^2(2\pi x)\,dx$

$a_n = \frac{2}{2}\int_0^2 \cos^2(2\pi x)\cos\left(\frac{n\pi x}{2}\right)dx$

Let's expand $f(x)$ using the double angle formula

$\cos(2\theta) = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{1}{2}\left[1 + \cos(2\theta)\right]$

$\theta = 2\pi x \qquad 2\theta = 4\pi x$

$f(x) = \frac{1}{2} + \frac{1}{2}\cos(4\pi x)$

Compare to F.S. $\qquad a_0 = \frac{1}{2} \qquad 4\pi x = \frac{n\pi x}{2}$

$a_8 = \frac{1}{2} \qquad n = 8$

$\frac{a_0}{2} + \sum^{\infty} a_n \cos(n\pi x$ …

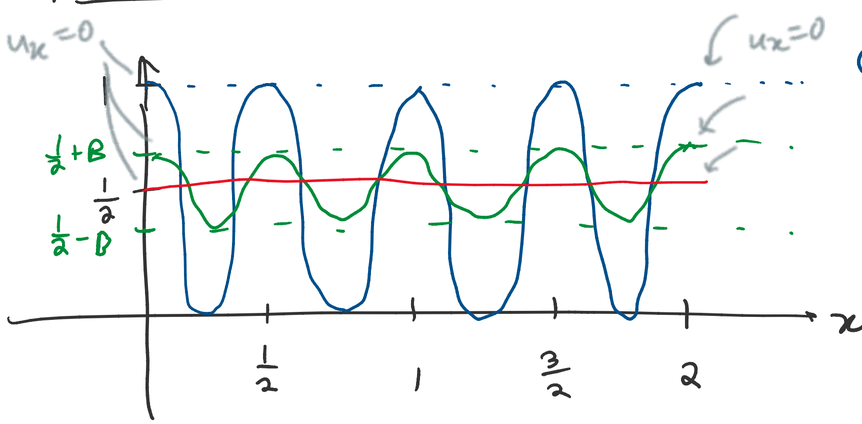
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$$

where $a_n = \begin{cases} 1 & \text{when } n=0 \\ \frac{1}{2} & \text{when } n=8 \\ 0 & \text{all other } n \end{cases}$

Just plug into our solution

$$u(x,t) = \frac{1}{2} + \frac{1}{2} e^{-\frac{1}{3}(8)^2\pi^2 t/4} \cos\left(\frac{8\pi x}{2}\right)$$

$$\boxed{u(x,t) = \frac{1}{2} + \frac{1}{2} e^{-16\pi^2 t/3} \cos(4\pi x)}$$



@ $t=0$

$u = \frac{1}{2} + \frac{1}{2}\cos(4\pi x)$

@ $t=1$

$u = \frac{1}{2} + \frac{e^{-16\pi^2/3}}{2} \cos(4\pi x)$

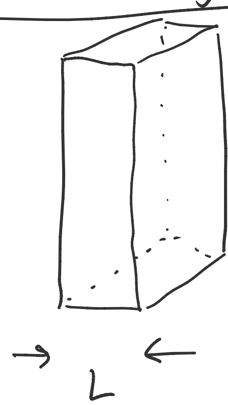
$= \frac{1}{2} + B\cos(4\pi x)$

as $t \to \infty$

$e^{-16\pi^2 t/3} \to 0$

$u \to \frac{1}{2}$

II. Heating a Slab:

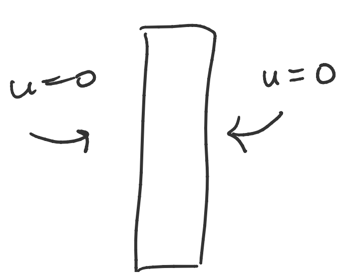


Assume material is homogeneous

inside heats uniformly

outside $\Delta T = 0$ (temp) for all time

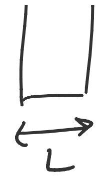
Look at a cross section



No heatgradient ↕

only heat gradient in ↔

use the same eqns to model



use the same eqns to model

Ex: A copper slab is 4 cm thick. Both faces are kept 0°C, but initally the interior is 100°C

(a) What is the temp of the center 3 s later?

(b) How long until the center cools to 5°C?

$L = 4\text{cm}$ $\quad f(x) = 100°C \quad k = 1.15\ \text{cm}^2/\text{s}$

$$\begin{cases} u_t = 1.15 u_{xx} \\ u(0,t) = u(4,t) = 0 \\ u(x,0) = 100°C \end{cases}$$

$\leftarrow$ homogeneous BC $\rightarrow$ sine series

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{4}\int_0^4 100 \sin\left(\frac{n\pi x}{4}\right)dx$$

$$= \frac{100}{2}\left[-\frac{\cos\left(\frac{n\pi x}{4}\right)}{\frac{n\pi}{4}}\right]_0^4$$

$$= \frac{100\cdot 4}{2n\pi}\left[-\cos(n\pi) + \cos(0)\right]$$

$$= \frac{200}{n\pi}\left[-(-1)^n + 1\right] = \begin{cases} \frac{400}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

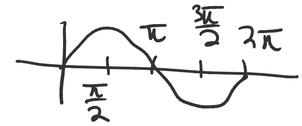
$$u(x,t) = \sum_{n \text{ odd}} \frac{400}{n\pi} e^{-1.15 n^2\pi^2 t/16} \sin\left(\frac{n\pi x}{4}\right)$$

(a) Temp at center $(x = 2)$ at $t = 3s$

(a) Temp at center ($x=2$) at $t = 3s$

$$u(2,3) = \sum_{n \text{ odd}} \frac{400}{n\pi} e^{-1.15\, n^2\pi^2 3/16} \sin\left(\frac{n\pi}{2}\right)$$

Write out the first 3 terms



$$\approx \frac{400}{(1)\pi} e^{-1.15(1)^2\pi^2\cdot 3/16} \cancelto{1}{\sin\left(\frac{\pi}{2}\right)}$$

$$+ \frac{400}{(3)\pi} e^{-1.15(3)^2\pi^2\cdot 3/16} \cancelto{(-1)}{\sin\left(\frac{3\pi}{2}\right)}$$

$$+ \frac{400}{(5)\pi} e^{-1.15(5)^2\pi^2\cdot 3/16} \cancelto{1}{\sin\left(\frac{5\pi}{2}\right)}$$

$$\approx 15.1591 - 2.04 \times 10^{-7} + \dots$$

$$\approx \boxed{15.16^\circ C}$$

(b) How long until center cools to $5^\circ C$?

To estimate let's use just $n=1$

$$5^\circ C = u(2,t) = \frac{400}{\pi} e^{-1.15\pi^2 t/16} \cancelto{1}{\sin\left(\frac{\pi}{2}\right)}$$

Solve for t

$$\ln\left(\frac{5\pi}{400}\right) = \ln\left(e^{-1.15\pi^2 t/16}\right)$$

$$\ln\left(\frac{5\pi}{400}\right) = -1.15\frac{\pi^2 t}{16}$$

$$t = \frac{-16 \ln(5\pi/400)}{1.15\pi^2} \approx \boxed{4.6s}$$