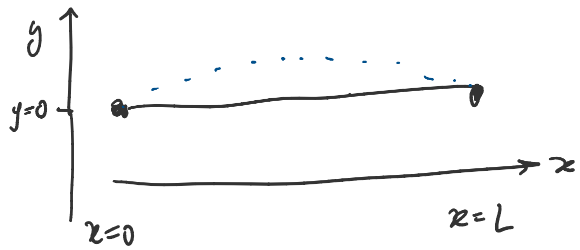
Lecture 21 (Sec 9.6)

☆ Vibrating Strings & the 1D Wave Equation



L string length

fixed ends

$y(x)$ - displacement from equilibrium

1D- Wave Equation

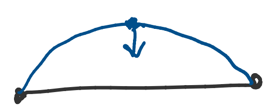
$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{or} \qquad y_{tt} = a^2 y_{xx}$$

where $a^2 = \dfrac{\text{tension}}{\text{density}} > 0$

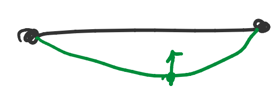
physical intuition

$y_{tt} \to$ acceleration of string

$y_{xx} \to$ curvature



$y_{xx} < 0$



$y_{xx} > 0$

$y_{tt} = a^2 y_{xx} \to$ means y_{tt} has the same sign as y_{xx}

$y_{tt} < 0$ acceleration downward

$y_{tt} > 0$ acceleration upwards

string wants to restore to equilibrium

BVP for vibrating string:

$$(*)\left\{\begin{array}{ll} y_{tt} = a^2 y_{xx} & 0 < x < L, \; t > 0 \\ y(0,t) = y(L,t) = 0 & \text{fixed endpoints} \end{array}\right.$$

... displacement

$$(*)\begin{cases} y_{tt} = a^2 y_{xx} \\ y(0,t) = y(L,t) = 0 & \text{fixed endpoints} \\ y(x,0) = f(x) & \text{initial displacement} \\ y_t(x,0) = g(x) & \text{initial velocity} \end{cases}$$

To solve — Separation of Variables

But first, let's simplify this problem by separating $(*)$ into 2 easier problems

Problem A	Problem B
$y_{tt} = a^2 y_{xx}$	$y_{tt} = a^2 y_{xx}$
$y(0,t) = y(L,t) = 0$	$y(0,t) = y(L,t) = 0$
$y(x,0) = f(x)$	$y(x,0) = 0$
$y_t(x,0) = 0$	$y_t(x,0) = g(x)$
solution $y_A(x,t)$	solution $y_B(x,t)$

Since the wave eqn is linear, we can write the general soln $(*)$ as $y(x,t) = y_A(x,t) + y_B(x,t)$

Solve Problem A first:

$$(A)\begin{cases} y_{tt} = a^2 y_{xx} \\ y(0,t) = y(L,t) = 0 \\ y(x,0) = f(x) \\ y_t(x,0) = 0 \end{cases}$$

Assumption $y(x,t) = X(x)\,T(t)$

$y_{tt} = X T''$ $\qquad$ $y_{xx} = X'' T$

$XT'' = a^2 X'' T$

$y_{tt} = X T''$

so $\quad XT'' = a^2 X'' T$

$$\frac{T''}{a^2 T} = \frac{X''}{X} = -\lambda \quad \left(\begin{array}{c}\text{separation}\\ \text{constant}\end{array}\right)$$

$X'' + \lambda X = 0$ $\qquad\qquad$ $T'' + a^2\lambda T = 0$

Endpoint BC

$y(0,t) = 0 = X(0)T(t) \rightarrow X(0) = 0$

$y(L,t) = 0 = X(L)T(t) \rightarrow X(L) = 0$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0 \end{cases}$$

same eqn for X as the heat equation

eigenvalues $\lambda_n = \frac{n^2\pi^2}{L^2}$

eigenfunctions $X_n = \sin\left(\frac{n\pi x}{L}\right)$ $\quad n = 1, 2, 3, \ldots$

Now, let's solve the T eqn

$T'' + a^2 \lambda T = 0 \qquad \lambda_n = \frac{n^2\pi^2}{L^2}$

$$T'' + \frac{a^2 n^2 \pi^2}{L^2} T = 0$$

BC: $\quad y(x,0) = f(x) = X(x)T(0)$

$y_t(x,0) = 0 = X(x)T'(0) \longrightarrow T'(0) = 0$

$$\begin{cases} T'' + \frac{a^2 n^2 \pi^2}{L^2} T = 0 \\ T'(0) = 0 \end{cases}$$

$$\Big\}\quad T'(0)=0$$

Solution:

$$T(t) = c_1 \cos\left(\frac{n\pi a t}{L}\right) + c_2 \sin\left(\frac{n\pi a t}{L}\right)$$

$$T'(t) = -\frac{n\pi a}{L} c_1 \sin\left(\frac{n\pi a t}{L}\right) + \frac{n\pi a}{L} c_2 \cos\left(\frac{n\pi a t}{L}\right)$$

$$T'(0) = \frac{n\pi a}{L} c_2 \cancel{\cos(0)}^{1} \Rightarrow c_2 = 0$$

So then eigenfunction $T_n = \cos\left(\frac{n\pi a t}{L}\right)$ for $n = 1, 2, 3, \ldots$

So the solution for each n

$$y_n(x,t) = \cos\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

General solution is:

$$\boxed{y_A(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)}$$

where BC: $y(x,0) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \qquad \text{Fourier sine series}$$

so $\boxed{A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}$

Now let's solve Problem B:

$$\begin{cases} y_{tt} = a^2 y_{xx} \\ y(0,t) = y(L,t) = 0 \\ y(x,0) = 0 \\ y_t(x,0) = g(x) \end{cases}$$

$$\begin{cases} y(x,0) = 0 \\ y_t(x,0) = g(x) \end{cases}$$

Again let's assume $y(x,t) = X(x)T(t)$

$$\frac{X''}{X} = \frac{T''}{aT} = -\lambda$$

The X solution is the same as Problem A

$$X_n = \sin\left(\frac{n\pi x}{L}\right) \qquad \lambda_n = \frac{n^2\pi^2}{L^2}$$

The T equation is:

$$\begin{cases} T'' + \frac{a^2n^2\pi^2}{L^2} T = 0 \\ T(0) = 0 \end{cases}$$

similarly, we find

$$T_n = \sin\left(\frac{n\pi a t}{L}\right)$$

$$\boxed{y_B(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)}$$

Apply BC $y_t(x,0) = g(x)$

$$\frac{\partial y_B}{\partial t} = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi a}{L} \cos\left(\frac{n\pi a t}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{\partial y_B}{\partial t}(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{L} \sin\left(\frac{n\pi x}{L}\right)$$ Fourier sine Series

$$\frac{B_n n\pi a}{L} = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\boxed{B_n = \frac{2}{n\pi a}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx}$$

The solution to $(*)$ $\quad y = y_A + y_B$

$$\boxed{\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \\ A_n &= \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \\ B_n &= \frac{2}{n\pi a}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx \end{aligned}}$$

<u>Ex</u>:

$$\begin{cases} y_{tt} = 4 y_{xx} & 0 < x < \pi, \quad t > 0 \\ y(0,t) = y(\pi,t) = 0 & \\ y(x,0) = \frac{1}{10}\sin(2x) & \leftarrow \text{Problem A} \\ y_t(x,0) = 0 & \end{cases}$$

Here $L = \pi$, $\quad f(x) = \frac{1}{10}\sin(2x)$, $g(x) = 0$, $\quad a = 2$

$B_n = 0$ since $g(x) = 0$

$$A_n = \frac{2}{\pi}\int_0^{\pi} \frac{1}{10}\sin(2x)\sin\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \frac{2}{10\pi}\int_0^{\pi} \sin(2x)\sin(nx)\,dx \quad \leftarrow \text{use orthogonality}$$

$$= \frac{2}{10\pi}\begin{cases} \frac{\pi}{2} & \text{if } n = 2 \\ 0 & \text{if } n \neq 2 \end{cases}$$

$A_2 = \frac{1}{10} \qquad A_n = 0$ for all $n \neq 2$

$$y(x,t) = A_2 \cos\left(\frac{2\pi 2 t}{\pi}\right)\sin\left(\frac{2\pi x}{2}\right)$$

$$\boxed{y(x,t) = \frac{1}{10}\cos(4t)\sin(2x)}$$

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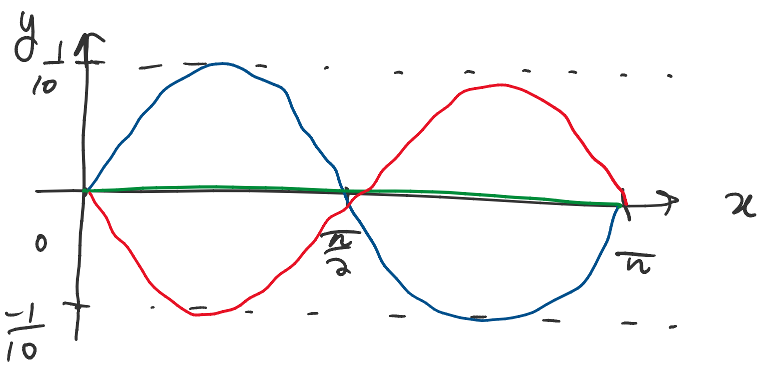

@ $t=0$, $y = \frac{1}{10}\sin(2x) = f(x)$

@ $t=\frac{\pi}{8}$ $y=0$

@ $t=\frac{\pi}{4}$ $y = \frac{-1}{10}\sin(2x)$

The amplitude $A(t) = \frac{1}{10}\cos(4t)$ oscillates between $\frac{-1}{10}$ and $\frac{+1}{10}$ with a period $p = \frac{\pi}{2}$ in time

Ex:

$$\begin{cases} y_{tt} = y_{xx} & 0<x<1, \quad t>0 \\ y(0,t) = y(1,t) = 0 \\ y(x,0) = -3 \\ y_t(x,0) = 5\sin(3\pi x) \end{cases}$$

Here $L=1$, $a=1$, $f(x)=-3$, $g(x) = 5\sin(3\pi x)$

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi t}{1}\right)\sin\left(\frac{n\pi x}{1}\right) + \sum_{n=1}^{\infty} B_n \sin(n\pi t)\sin(n\pi x)$$

where

$$A_n = \frac{2}{1}\int_0^1 (-3)\sin(n\pi x)\,dx$$

$$= -6\left[-\frac{\cos(n\pi x)}{n\pi}\right]_0^1$$

$$= \frac{6}{n\pi}\left[\cos(n\pi) - \cos(0)\right] = \frac{6}{n\pi}\left[(-1)^n - 1\right]$$

$$= \begin{cases} \frac{-12}{n\pi} & \text{if } n \text{ odd} \end{cases}$$

$$= \begin{cases} \frac{-12}{n\pi} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

where

$$B_n = \frac{2}{n\pi(1)} \int_0^1 5\sin(3\pi x)\sin(n\pi x)\,dx$$

$u = \pi x$
$du = \pi\, dx$

$$= \frac{10}{n\pi^2} \int_0^\pi \sin(3u)\sin(nu)\,du$$

orthogonality

$$= \frac{10}{n\pi^2} \begin{cases} \frac{\pi}{2} & n = 3 \\ 0 & n \neq 3 \end{cases}$$

$$B_3 = \frac{10\cancel{\pi}}{2n\pi^{\cancel{2}}} = \frac{5}{n\pi}$$

$B_n = 0$ if $n \neq 3$

So the full solution is

$$\boxed{y(x,t) = \frac{-12}{\pi} \sum_{n \text{ odd}} \frac{1}{n}\cos(n\pi t)\sin(n\pi x) + \frac{5}{n\pi}\sin(3\pi t)\sin(3\pi x)}$$