Lecture 22 (Sec 9.7)

★ Steady - State Temperature & Laplace's Equation

I. motivation:

1D Heat Eqn: $u_t = k u_{xx}$

1D Wave Eqn: $u_{tt} = a^2 u_{xx}$

We want to extend this to 2D (x and y)

2D Heat Eqn: $u_t = k(u_{xx} + u_{yy})$

2D Wave Eqn: $u_{tt} = a^2 \underbrace{(u_{xx} + u_{yy})}$

$$\nabla^2 u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u$$

Laplacian

<u>Goal</u>: want to know the steady state solution

($u_t = 0$ or $u_{tt} = 0$)

solution no longer changing in time

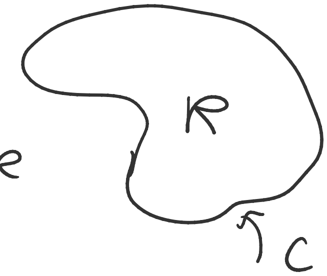
2D Heat Eqn steady state $\quad u_t = 0 = k \nabla^2 u$

2D Wave Eqn steady state $\quad u_{tt} = 0 = a^2 \nabla^2 u$

steady state when $\boxed{\nabla^2 u = 0}$

<u>Laplace's Equation</u>

II. Dirichlet Problems

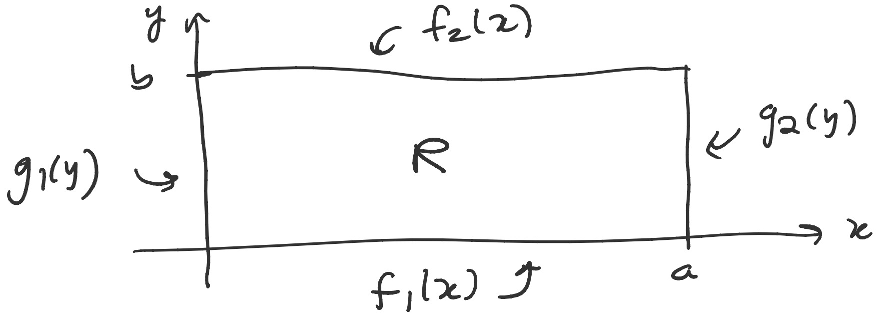


$$\begin{cases} \nabla^2 u = u_{xx} + u_{yy} = 0 & \text{on region } R \\ u(x,y) = f(x,y) & \text{on curve } C \end{cases}$$

This is called a <u>Dirichlet Problem</u>

If $f(x,y)$ and the curve C are "nice" then there is a unique solution $u(x,y)$

Let R be a rectangle



$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x,0) = f_1(x) \\ u(x,b) = f_2(x) \\ u(0,y) = g_1(y) \\ u(a,y) = g_2(y) \end{cases}$$

To solve this, use separation of variables.

<u>Ex</u>:
$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0,y) = u(a,y) = u(x,b) = 0 \\ u(x,0) = f(x) \end{cases}$$

Here $g_1 = g_2 = f_2 = 0$ $\qquad f_1(x) = f(x)$

Separation of Variables

Assume $u(x,y) = X(x)\,Y(y)$

$u_{xx} = X''Y \qquad u_{yy} = XY''$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$ ← separation constant

$$\frac{X''}{X} = -\lambda \qquad\qquad -\frac{Y''}{Y} = -\lambda$$

$$\frac{X''}{X} = -\lambda \qquad\qquad -\frac{1}{Y} = -\lambda$$

$$X'' + \lambda X = 0 \qquad\qquad Y'' - \lambda Y = 0$$

Evaluate the BC

$u(0,y) = 0 = X(0)Y(y) \longrightarrow X(0) = 0$

$u(a,y) = 0 = X(a)Y(y) \longrightarrow X(a) = 0$

$u(x,b) = 0 = X(x)Y(b) \longrightarrow Y(b) = 0$

$u(x,0) = f(x) = X(x)Y(b)$

Solve X eqn first

$X'' + \lambda X = 0 \qquad X(0) = X(a) = 0$

$X_n = \sin\left(\frac{n\pi x}{a}\right)$ eigenfunction $\qquad \lambda_n = \frac{n^2\pi^2}{a^2}$ e eigenvalue

Solve the Y eqn

$Y'' - \lambda Y = 0 \qquad Y(b) = 0 \qquad \lambda_n = \frac{n^2\pi^2}{a^2}$

$Y_n'' - \frac{n^2\pi^2}{a^2} Y_n = 0$ since this coeff is neg soln is cosh and sinh

$$Y_n = c_1 \cosh\left(\frac{n\pi y}{a}\right) + c_2 \sinh\left(\frac{n\pi y}{a}\right)$$

$$Y_n(b) = 0 = c_1 \cosh\left(\frac{n\pi b}{a}\right) + c_2 \sinh\left(\frac{n\pi b}{a}\right)$$

$$c_2 = -\frac{c_1 \cosh\left(\frac{n\pi b}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}$$

so $$Y_n(y) = c_1 \cosh\left(\frac{n\pi y}{a}\right) - \frac{c_1 \cosh\left(\frac{n\pi b}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)} \sinh\left(\frac{n\pi y}{a}\right)$$

$$= \frac{c_1}{\sinh\left(\frac{n\pi b}{a}\right)} \underbrace{\left[\cosh\left(\frac{n\pi y}{a}\right)\sinh\left(\frac{n\pi b}{a}\right) - \cosh\left(\frac{n\pi b}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)\right]}$$

use a subtraction formula for hyperbolic trig fcns

$$\sinh(\alpha-\beta) = \sinh(\alpha)\cosh(\beta) - \cosh(\alpha)\sinh(\beta)$$

Here $\alpha = \frac{n\pi b}{a}$ $\quad \beta = \frac{n\pi y}{a}$

$$Y_n(y) = \underbrace{\frac{c_1}{\sinh\left(\frac{n\pi b}{a}\right)}}_{c_n} \sinh\left(\frac{n\pi}{a}(b-y)\right)$$

$$Y_n(y) = c_n \sinh\left(\frac{n\pi(b-y)}{a}\right)$$

So full solution

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi(b-y)}{a}\right)\sin\left(\frac{n\pi x}{a}\right)}$$

Now, let's use the last BC

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} \left[c_n \sinh\left(\frac{n\pi b}{a}\right)\right]\sin\left(\frac{n\pi x}{a}\right)$$

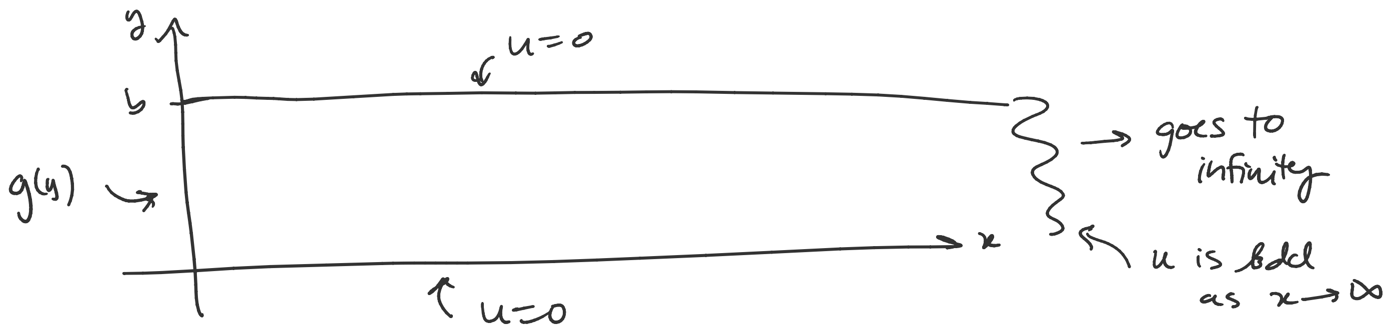
Fourier sine series

$$c_n \sinh\left(\frac{n\pi b}{a}\right) = b_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

$$\boxed{c_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx}$$

III. Infinite strip:

Let R be the "semi-infinite" strip



$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } R \\ u(x,0) = u(x,b) = 0 & 0 < x < \infty \\ u(x,y) \text{ is bounded as } x \to +\infty & \\ u(0,y) = g(y) & \end{cases}$$

Separation of variables:

$$u(x,y) = X(x)\,Y(y)$$

Look at BC

$$u(x,0) = 0 = X(x)Y(0) \longrightarrow Y(0) = 0$$

$$u(x,b) = 0 = X(x)Y(b) \longrightarrow Y(b) = 0$$

So write

since Y has endpoint conditions

$$\frac{Y''}{Y} = -\frac{X''}{X} = -\lambda$$

$$Y'' + \lambda Y = 0$$

$$Y(0) = Y(b) = 0$$

$$\lambda = \frac{n^2\pi^2}{b} \quad \text{eigenvalue}$$

$$Y_n = \sin\left(\frac{n\pi y}{b}\right) \quad \text{eigenfcn}$$

$$X'' - \lambda X = 0$$

Solve the X eqn.

Solve the X eqn.

$$X_n'' - \frac{n^2\pi^2}{b^2} X_n = 0$$

We can write either

$$X_n = C_1 \cosh\left(\frac{n\pi x}{b}\right) + C_2 \sinh\left(\frac{n\pi x}{b}\right)$$

or we can write

$$\boxed{X_n = D_1 \exp\left(\frac{n\pi x}{b}\right) + D_2 \exp\left(-\frac{n\pi x}{b}\right)}$$

(since $\cosh(z) = \frac{e^{z}+e^{-z}}{2}$ $\quad \sinh(z) = \frac{e^{z}-e^{-z}}{2}$)

Choose the exponential form because its easier to apply the boundedness condition

$\exp\left(-\frac{n\pi x}{b}\right)$ is bounded as $x \to \infty$ $\quad (D_1 = 0)$

so $X_n = \exp\left(-\frac{n\pi x}{b}\right)$

so the solution

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)}$$

use the last BC

$$u(0,y) = g(y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi y}{b}\right) \leftarrow \text{Fourier sine series}$$

$$\boxed{b_n = \frac{2}{b}\int_0^b g(y) \sin\left(\frac{n\pi y}{b}\right) dy}$$

IV. Circular Domain.

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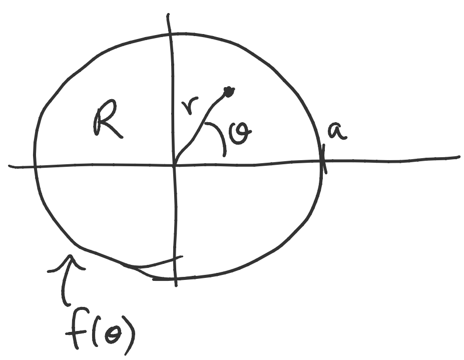

use polar coordinates (r, θ)

Laplace's Equation in polar coordinates

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \\ u(a, \theta) = f(\theta) \\ u(r,\theta) = u(r, \theta + 2\pi) \quad \text{– periodicity} \\ \text{continuous at } r = 0 \end{cases}$$

Solve using Separation of Variables

$$u(r,\theta) = R(r)\,\Theta(\theta)$$

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

$$\frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

$$r^2 R'' + rR' - \lambda R = 0 \qquad\qquad \Theta'' + \lambda\Theta = 0$$

Solve the Θ eqn first

$$\Theta'' + \lambda\Theta = 0$$

$$\Theta(\theta) = A\cos(\sqrt{\lambda}\,\theta) + B\sin(\sqrt{\lambda}\,\theta)$$

Apply the periodicity condition

$$u(r,\theta) = u(r, \theta + 2\pi)$$

$$\Theta(\theta) = \Theta(\theta + 2\pi)$$

$$\cos(\sqrt{\lambda}\,\theta) = \cos(\sqrt{\lambda}(\theta + 2\pi))$$

$$\sqrt{\lambda}\,2\pi = n\,2\pi$$
$$\lambda = n^2$$

$$\Theta_n(\theta) = A\cos(n\theta) + B\sin(n\theta)$$

Note : its possible $\lambda = 0$

$$\Theta'' = 0$$
$$\Theta = A + \cancel{B\theta} \text{ not periodic } B = 0$$

when $\lambda = 0$ $\quad \Theta_0 = \frac{a_0}{2}$

Now lets solve the R equation $\qquad \lambda = n^2$

$$r^2 R_n'' + r R_n' - n^2 R_n = 0$$

Try $R_n = r^k$ for some integer k

$$R_n' = k r^{k-1} \qquad R_n'' = k(k-1) r^{k-2}$$

$$r^2 k(k-1) r^{k-2} + r k r^{k-1} - n^2 r^k = 0$$
$$\left(k(k-1) + k - n^2 \right) r^k = 0$$

$$k^2 \cancel{- k + k} - n^2 = 0$$
$$k^2 = n^2$$
$$k = \pm n$$

$$R_n(r) = C_n r^n + \frac{D_n}{r^n} \qquad \left(\text{when } n = 0 \quad R_0 = C_0 \right)$$

continuity at $r = 0$

$$R_n(0) = C_n(0) + \cancel{\frac{D_n}{0}} \qquad D_n = 0$$

$$R_n(r) = C_n r^n$$

∞

$K_n(r) = r^n$

$$\boxed{u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) r^n}$$

use the last BC

$$u(a,\theta) = f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) a^n$$

so $$a^n a_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta)\, d\theta$$

$$\boxed{a_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \cos(n\theta)\, d\theta}$$

Similarly

$$\boxed{b_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \sin(n\theta)\, d\theta}$$

$$\boxed{a_0 = \frac{1}{\pi} \int_0^{2\pi} f(\theta)\, d\theta}$$